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DETERMINATION OF THE GRAVITATIONAL CONSTANT BY ATOM INTERFEROMETRY

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Abstract

This work describes the realization of an atom interferometry–based gravity gradiometer aiming at an accurate determination of the Newtonian gravitational constant $G$. The scheme is conceptually different from all the measurements published so far. It relies on the simultaneous measurement of the accelerations of two vertically displaced, freely falling cold atomic samples, by means of Raman atom interferometry. The use of two displaced sensors allows a dramatic common mode phase–noise suppression, so highly accurate gradiometric measurements are possible.

Well–characterized source masses are placed close to the interferometers in two different positions and the relative phase shift is recorded. This removes the effect of all the fixed masses and makes the measurement doubly–differential: both in time and space. A faithful simulation of the gravitational field of the source masses provides a phase shift as a function of $G$. This is compared to the acquired differential phase shift and a value of the gravitational constant is extracted. A study of the systematics leads to an aimed accuracy of the order of $10^{-4}$.

The gravity–gradiometer is presently working. Measurement runs of continuous data acquisition are now possible. Preliminary $G$ measurements at the 1% level have been performed.
Chapter 1

Introduction

1.1 Past measurements of the Newtonian gravitational constant

The first discovery of one of the four fundamental forces is represented by the famous Newtonian law of gravitation

\[ F = \frac{GMm}{r^2} \]

The coupling constant that governs the gravitational attraction between two massive bodies is the worst known fundamental physical constant [1]. Improvements on the knowledge of \( G \) would not only satisfy challenging souls and metrologists, but would be useful in various fields.

Several models described in theoretical physics contain the gravitational constant as an essential parameter and its uncertainty limits the possible predictions. A much more accurate knowledge of \( G \) could discriminate among contrasting theories giving credit to some and not to others. Recently, thanks to some technological advances, many efforts have been devoted to study gravity at short distances looking for deviations from the Newtonian law. Evidences of a deviation on the micrometer scale would prove the existence of a precise number of compactified extra-dimensions \([2, 3]\) in which gravity, but not the other fundamental forces, acts. This would be a possible explanation to the apparent weakness of the gravitational coupling constant.

Astrophysics is evidently linked to the gravitational constant since it governs the motion of the planets and the dynamic evolution of the universe. A better knowledge of the value of \( G \) and of its possible variation in time \([4]\) would directly lead to a better precision on many celestial system parameters.
and information about the past and the future of the universe. Geocentric $(GM_\oplus)$ and heliocentric $(GM_\odot)$ gravitational constants, for example, have been measured with an accuracy of a few ppb. This could be an easy way to determine Earth and Sun mass once $G$ is better known. Also celestial bodies internal evolution strongly depends on $G$. For example the stars luminosity can be expressed [5] by the mass–luminosity relation as $L = G^2 M^5$.

All the Earth sciences like geophysics and geodesy depend on the precise knowledge of the gravitational constant in their geoid calculation [6], seismic studies, reconstruction of Earth’s crust, plate tectonics and volcanos activities.

The CODATA–2002 recommended value for $G$ is $6.6742(10) \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$ [4]. Despite the large number of experiments [7, 8, 9] realized since 1798, when Cavendish performed his gravity measurements, $G$ is still affected by an uncertainty larger than 150 ppm, much higher than that of any other fundamental constant.

Starting with the first measurements at the 1% level [10], declared uncertainties were gradually reduced to 10 ppm [11]. Even considering the
most recent ones though [12, 13, 14, 11, 15, 16, 17, 18], the results\(^1\) differ by several hundreds ppm (see figure 1.1). Some groups around the world [19] are presently working on different experiments aiming to reach a precision of \(10^{-5}\).

Among the possible sources for this unexplained discrepancy two in particular appear mostly critical. First, the gravitational force is much weaker than the other fundamental forces and can not be shielded, therefore a dense source mass is needed to induce a detectable attraction on the probe mass and a differential measurement scheme becomes essential for an accurate measurement. The knowledge of \(G\) directly depends on the precision and accuracy to which the induced field is known, so the position of the source masses and their density distribution are crucial parameters, but it can be difficult to accurately characterize them. Second, the majority of the experiments performed so far are based on macroscopic suspended masses; systematic effects and parasitic couplings in suspending fibers are not well understood and could be responsible for the observed discrepancies.

Getting rid of all the systematic problems related with suspensions Schwarz \textit{et al.} (1999) performed a \(G\) measurement [20] realizing an optical Michelson interferometer with a free–falling corner cube on one interferometer arm. The uncertainty level reached was though not better than 1500 ppm. The macroscopic dimensions of the mirror and its not precisely controllable behaviour in the presence of electric and magnetic fields could in principle be responsible for other effects interpreted as gravitational attraction.

The idea of using microscopic test masses as neutral atoms to probe the gravitational field generated by a well–characterized source mass solves also the problem to control the position and the physical properties of the probe masses. New measurement schemes based on different technologies are always welcome when attempting to determine a physical constant. The comparison between a large variety of experiments helps in fact to understand systematics and aims at a more accurate determination of the constant.

\(^1\)The values reported on the graph for the measurements of Gundlach & Merkowitz (2002) and Luo (2003) are not the published values, but those subsequently corrected and considered for the determination of the CODATA 2002 value.
1.2 Cold atom interferometers as highly sensitive inertial detectors

The possibility to realize a gravity experiment with atoms became feasible thanks to the great improvements in understanding the atomic structure and its physical properties and to the enormous and rapid development of manipulation techniques following the revolutionary discovery of quantum mechanics and of the wave–like behaviour of matter.

Enrico Fermi realized the first matter–wave interferometer in 1947. In his experiment [21] slow neutrons were differently Bragg diffracted by crystal planes with a different chemical composition and the relative sign of the neutron scattering length from different nuclei was measured.

A few years later the first experiment on matter–wave interferometry with electrons [22] was demonstrated. It was a Mach–Zehnder type interferometer with crystals of a few hundreds atomic layers thickness used as mirrors.

Already at that time it was clear that neutral matter was better than charged particles or photons for realizing highly sensitive devices. It is much less sensitive to perturbing electric and magnetic fields than charged particles and its typical speed can be much lower than c allowing a longer interaction time within a fixed length scale. Neutrons though were difficult to produce in the lab since accelerators were needed and the dream of observing interference patterns on much more complex and massive systems like entire atoms became feasible only in the 90’s when several groups around the world demonstrated various atom interferometers.

Many interferometer schemes were realized [23, 24], some using material objects as splitters and mirrors [25, 26, 27], some other making use of light [28, 29] and thus inverting the roles that matter and light have in classical optical interferometers.

In recent years the experimental techniques to manipulate atoms have improved significantly leading to a much higher coherence level of the atoms with laser–cooling first and then achieving Bose–Einstein condensation, but in particular the interrogation time could rise by orders of magnitude and experiments in laboratories became more and more feasible and able to reach higher sensitivity.

Different interferometer schemes based on cold atoms were demonstrated. Standing waves orthogonal to the atomic beam direction were used to produce spatially diverging paths through Bragg diffraction [30], two–photon
Raman transitions in Mach–Zehnder type interferometer schemes allowed large momentum transfer [28] and optical lattices were used on ultra cold atoms and on condensates to observe Bloch oscillations [31, 32, 33, 34]. Furthermore the state-of-the-art atomic clocks consist in a cold atomic fountain with a microwave cavity and they can be considered as interferometers as well [35, 36].

In particular cold atoms interferometers are at least as sensitive as the best conventional devices in the field of inertial sensors: Earth rotation rate was measured with a short-term sensitivity of $6 \times 10^{-10}$ rad/s over 1 s of integration [37, 38], a gravimeter with a resolution of $2 \times 10^{-8}$ g after a single 1.3 s measurement cycle and $1 \times 10^{-10}$ g after two days of integration time was realized [39, 40]. The Earth gravity gradient was also measured with an uncertainty of 5% [41, 42].

Engineered transportable versions of these sensors [43, 44] could be applied in fields ranging from making real-time acquisition of airplane motion and navigation, mapping Earth mass anomalies and studying the surface properties, monitoring volcanos activities and gravitational changes before earthquakes. Furthermore atom interferometers have been used to explore fundamental physics by measuring fundamental constants with high precision like $h/m_{Cs}$ [45], $h/m_{Na}$ [46] and atom polarizability [29, 30] and by testing the equivalence principle [39, 47], $1/r^2$ Newton’s law at short distances [34] and general relativity [48]. A first demonstration of the detection of the gravitational field generated by lead test masses was also obtained at Yale with the apparatus for the Earth gradient determination [49, 50].

1.3 The MAGIA experiment

In the MAGIA\textsuperscript{2} experiment [51, 52], described in this thesis, microscopic test masses represented by $^{87}$Rb atoms probe the gravitational field generated by nearby well-characterized tungsten source masses.

Clouds of atoms are laser-cooled in a magneto-optical trap and launched upwards in an atomic fountain. Ensembles of about $10^6$ atoms in a well defined internal state, weakly sensitive to external electromagnetic fields, with a narrow vertical distribution and a vertical temperature below the recoil temperature are selected during the free–fall motion.

\textsuperscript{2}MAGIA is the name of the INFN (Istituto Nazionale di Fisica Nucleare) project and stands for Accurate Measurement of $G$ by Atom Interferometry.
Each atom can be described here as a two-level system with the internal state coupled to the external momentum. A vertical interferometer is realized using atoms as matter-waves and light pulses as optical elements. The first pulse splits the external and internal atomic wavefunction into two parts that independently evolve. The second pulse acts as a mirror inverting the internal state and provoking a momentum exchange. The last pulse recombines the two matter-waves creating interference. Relative to the light wavefronts the atomic position at the time of the three pulses is registered directly into the atomic wavefunction. The probability to measure the atom in one state or in the other depends on the different internal evolution along the two interferometer paths. Atoms are very sensitive to inertial accelerations and rotations, each of them contributing to the final measured phase shift. The experiment was designed to minimize most unwanted spurious effects through a differential measurement. The dominant phase term measured with this scheme on a single cloud of atoms is in fact induced by local gravity.

Two different samples are then launched in the fountain in such a way that they are vertically separated, but travel with the same velocity at the same time. The interferometer pulses are simultaneously applied on both clouds and the acquired differential phase is related to the vertical gravity gradient.

Dense, homogeneous and precisely machined tungsten source masses (516 kg) are placed around the atomic fountain, arranged in two independent sets of 12 cylinders each. The gradiometric measurement is repeated, leaving the atomic launch parameters unchanged, for two source masses configurations, as schematically drawn in figure 1.2, with a careful choice of the parameters in order to minimize systematic errors. In particular the two masses configurations are chosen in such a way that the resulting induced differential phase in each measurement can be as insensitive as possible to the atomic position.

The double differential measurement cancels out spatial and temporal disturbing effects like Earth gravity, its gradient and its temporal changes, uniform fields and constant electric or magnetic field gradients, basically leaving only the term induced by the gravitational field generated by the moving source masses. The comparison between the measured phase shift and the one obtained from a computer simulation leads to the determination of the gravitational constant $G$.

To estimate the achievable precision, one can consider the atom gra-
diometer state-of-the-art [42] sensitivity to differential accelerations \((4 \times 10^{-9} \text{ g Hz}^{-1/2})\) and compare it to the differential acceleration signal \((1.4 \times 10^{-7} \text{ g})\) induced by the source masses on the two atomic sensors in our experiment. This would yield a sensitivity of about \(6 \times 10^{-2} \text{ G Hz}^{-1/2}\). An integration over four days measurement time leads to an uncertainty of 100 ppm. An error budget analysis of the experimental parameters projects the accuracy on the final determination of \(G\) to below 100 ppm as well.
1.4 Thesis overview

The MAGIA experiment is described in this thesis taking into account theoretical aspects, but mainly from an experimental point of view.

First of all a brief theoretical introduction on the basic tools used to perform atom–light interaction and realize inertial sensitive interferometers is presented in chapter 2. An overview of the terms contributing to the phase shift, useful for systematic studies, is reported at the end.

Chapter 3 contains a detailed description of the experimental apparatus: the laser system and the way the several light beams are provided in terms of frequency, intensity, polarization and dimensions; the vacuum system with motivations for the choice of the geometrical shape and of the materials; the source masses, their precise holder and elevator and their characterization tests performed; the real-time remote control of the experiment.

The experimental sequence starting from atom trapping and cooling to their detection and signal extraction after the interferometer during the free fall is described in chapter 4. Preliminary measurements performed on the apparatus for a better characterization of the final $G$ measurement are reported.

Chapter 5 contains the details of the first $G$ measurements. Gravitational field simulation methods, optimization of the parameters choice and experimental procedures are reported.

The thesis ends with a summary of the obtained results and the future prospects for the experiment in chapter 6.
Chapter 2

Raman atom interferometry

This chapter presents some basic theoretical tools useful to introduce Raman atom interferometry. After a brief comparison between optical and atom interferometers, focusing on the impressive analogies between the two schemes and on the matter-wave double nature of atoms and light, atom interferometers will be studied in detail. The starting point will be the simple case of a two-level atom illuminated by monochromatic light. Then stimulated Raman transitions will be introduced using a picture with a multi-level atom and two counterpropagating light beams. Experimental parameters and variables, that will be used in this thesis, are introduced in order to make the reading easier and more directly comprehensible. The main physical phenomena that can induce a phase shift in the interferometer signal are separately considered and analytical expressions for the different phase terms are derived.

2.1 Light interferometry – atom interferometry

In an optical interferometer a coherent light beam is split into two beams that travel through different paths, they are then recombined and an interference pattern can be observed when varying the relative path length. Material objects are used to split, reflect and recombine the light wave. There are several demonstrated optical interferometer schemes [53] using different components like slits, gratings, beam splitters, etc.; in the following I will focus only on one particular kind, considering the analogies with the experiment described in this thesis.

In a Mach–Zehnder type interferometer one can use a glass beam splitter
for splitting and recombining and mirrors for reflecting the light waves. In figure 2.1, on the left side, a schematic diagram of an optical Mach–Zehnder type interferometer is drawn. As long as the two optical paths are equal no phase shift is observed. An extra element can be introduced on one of the two paths, changing its optical length and thus introducing a phase shift $\Delta \phi$ proportional to the additional path length.

By now, it is well-known and it has been demonstrated in a large variety of experiments that light and matter particles have both a dual nature and behave in similar ways, depending on the experimental configuration, either as quantum particles (photons–massive particles) or as waves (light waves–matter waves).

An atom interferometer is an elegant example demonstrating this similar nature. In fact, with evident analogies as also illustrated in figure 2.1, it can be described in the following way: let us consider a two-level atom (with levels $|1\rangle$ and $|2\rangle$), freely moving in a certain direction. A light pulse of radiation that couples the two levels is sent on the atom whose wavefunction gets split into two parts with equal amplitudes, both spatially and in terms of the internal state. These two parts of the wavefunction travel through different paths and experience different local forces. When they are recombined, still with light pulses, an interference pattern can be observed on the internal state population as a function of the relative phase accumulated on the two different atomic paths.

Atoms are intrinsically more sensitive than photons to environmental conditions such as magnetic and electric fields and, being massive particles, also to gravitational fields. They have a much more complex internal structure that allows more degrees of freedom and several possibilities to manipulate them. For these reasons one can realize high sensitivity external field sensors based on atoms.
2.2 Atom–light field interaction

2.2.1 Single photon transition on a two–level atomic system

Let us first consider a simple system in which a two–level atom, with energy eigenvalues $\hbar \omega_a$ and $\hbar \omega_b$ ($\omega_b - \omega_a = \omega_{ab}$), interacts with a monochromatic electric field with frequency $\omega_L$ ($\omega_L - \omega_{ab} = \delta$)

$$E_L(x, t) = E_0 \cos[k_L \cdot x - \omega_L t + \phi_L]$$  \hspace{1cm} (2.1)

where $x=(x, y, z)$ is the position vector of the atom and $k_L$ the field wavevector.

A quantum approach [34, 23] will be used taking into account both internal and external atomic degrees of freedom. An appropriate basis choice consists in a continuous one for the momentum ($p$) and a quantized one for the internal atomic eigenstates, $|a\rangle$ for the ground state and $|b\rangle$ for the excited one.

If $p$ and $d$ are the atomic external momentum and electric dipole moment operators, the Hamiltonian $\mathcal{H}$ of the system, in the dipole approximation, can be written as

$$\mathcal{H} = \frac{p^2}{2m} + \hbar \omega_a |a\rangle \langle a| + \hbar \omega_b |b\rangle \langle b| - d \cdot E_L. \hspace{1cm} (2.2)$$

In this frame it is possible to express an atomic state as tensor product between the two Hilbert spaces

$$|a, p_1\rangle = |a\rangle \otimes |p_1\rangle \hspace{1cm} |b, p_2\rangle = |b\rangle \otimes |p_2\rangle. \hspace{1cm} (2.3)$$

From (2.1) and (2.2) and considering that

$$e^{\pm ik_L \cdot x} = \int d^3p e^{\pm ik_L \cdot x} |p\rangle \langle p| = \int d^3p |p \pm \hbar k_L\rangle \langle p| \hspace{1cm} (2.4)$$

it is possible to state that, together with photon absorption or stimulated emission, the external atomic momentum changes by a discrete amount, $\pm \hbar k_L$. There is indeed a direct coupling between internal state and external momentum of the atomic system and its temporal evolution can be described using only two eigenstates:

$$|1\rangle = |a, p\rangle \hspace{1cm} E_1 = \hbar \omega_a + \frac{|p|^2}{2m} = \hbar \omega_1$$

$$|2\rangle = |b, p + \hbar k_L\rangle \hspace{1cm} E_2 = \hbar \omega_b + \frac{|p + \hbar k_L|^2}{2m} = \hbar \omega_2$$
It is convenient to define the relevant frequencies in this way

\[ \omega_0 = \omega_2 - \omega_1 = \omega_{ab} + \frac{p \cdot k_L}{m} + \frac{\hbar |k_L|^2}{2m} \]  \quad (2.5)  

\[ \Delta = \omega_L - \omega_0 = \omega_L - (\omega_{ab} + \frac{p \cdot k_L}{m} + \frac{\hbar |k_L|^2}{2m}) \]  \quad (2.6)  
clearly showing the Doppler effect and the recoil terms. The atomic wavefunction is then a linear superposition of the two states indicated above

\[ |\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle \]  \quad (2.7)  

where \(a_1(t)\) and \(a_2(t)\) are the amplitude probabilities of detecting the atom in the corresponding state at time \(t\). Let us now split each \(a_i(t)\) in two terms, one due to the interaction with the electric field and another one related with the free evolution of the atomic wavefunction

\[ a_1(t) = c_1(t)e^{-i\omega_1 t} \quad a_2(t) = c_2(t)e^{-i\omega_2 t}. \]  \quad (2.8)  

The atomic state can then be written in the form

\[ |\psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle. \]  \quad (2.9)  

From the Schrödinger equation

\[ i\hbar \frac{d}{dt} |\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle \]  \quad (2.10)  

with the Hamiltonian now defined as

\[ \mathcal{H} = \hbar \omega_1 |1\rangle \langle 1| + \hbar \omega_2 |2\rangle \langle 2| - d \cdot E_L \]  \quad (2.11)  

in the rotating wave approximation [55], one gets the following differential equations for the coefficients \(c(t)\)

\[ ic_1(t) = \frac{\Omega}{2} e^{i\Delta t} e^{-i\phi_L} c_2(t) \]  \quad (2.12)  

\[ ic_2(t) = \frac{\Omega^*}{2} e^{-i\Delta t} e^{-i\phi_L} c_1(t) \]  \quad (2.13)  

where \(\Omega\) is the Rabi frequency defined as

\[ \Omega = -\frac{\langle a|d \cdot E_0|b\rangle}{\hbar}. \]  \quad (2.14)
Introducing the generalized Rabi frequency $\Omega'$ as
\[ \Omega' = \sqrt{\Delta^2 + \Omega^2} \] (2.15)
the exact solution for the coefficients becomes
\[
c_1(t) = e^{i\frac{\Omega t}{2}} \left\{ c_1(0) \left[ \cos \left( \frac{\Omega t}{2} \right) - i \frac{\Delta}{\Omega'} \sin \left( \frac{\Omega t}{2} \right) \right] + c_2(0) e^{-i\phi_L} \left[ -i \frac{\Omega}{\Omega'} \sin \left( \frac{\Omega t}{2} \right) \right] \right\} + c_2(0) e^{-i\frac{\Delta t}{2}} \left\{ c_1(0) \cos \left( \frac{\Omega t}{2} \right) - i c_2(0) e^{-i\phi_L} \sin \left( \frac{\Omega t}{2} \right) \right\},
\] (2.16)
\[
c_2(t) = e^{-i\frac{\Delta t}{2}} \left\{ c_1(0) e^{i\phi_L} \left[ -i \frac{\Omega}{\Omega'} \sin \left( \frac{\Omega t}{2} \right) \right] + c_2(0) \left[ \cos \left( \frac{\Omega t}{2} \right) + i \frac{\Delta}{\Omega'} \sin \left( \frac{\Omega t}{2} \right) \right] \right\}.
\] (2.17)

By performing a measurement of the atomic state as a function of the interaction time $\tau$ one would find that the probability of finding the atom in the ground state $P_1$ or in the other $P_2$ periodically oscillates at frequency $\Omega'$
\[
P_1(\tau) = \left( \frac{\Omega}{\Omega'} \right)^2 \frac{1 + \cos \Omega' \tau}{2} \quad P_2(\tau) = \left( \frac{\Omega}{\Omega'} \right)^2 \frac{1 - \cos \Omega' \tau}{2}.
\] (2.18)

In the particular case of perfect frequency matching ($\Delta = 0$) we have
\[
c_1(t) = c_1(0) \cos \left( \frac{\Omega t}{2} \right) - ic_2(0) e^{-i\phi_L} \sin \left( \frac{\Omega t}{2} \right)
\] (2.19)
\[
c_2(t) = -ic_1(0) e^{i\phi_L} \sin \left( \frac{\Omega t}{2} \right) + c_2(0) \cos \left( \frac{\Omega t}{2} \right).
\] (2.20)
In such conditions, if a light pulse of temporal length \( \tau \) is sent on an atom initially in the state \( |1\rangle \) its final state at the end of the interaction will be

\[
|\psi(\tau)\rangle = \cos \left( \frac{\Omega \tau}{2} \right) |1\rangle + e^{-i\frac{\tau}{2} \phi_L} \sin \left( \frac{\Omega \tau}{2} \right) |2\rangle.
\]  

(2.21)

By plotting \( P_1 \) as a function of the interaction time \( \tau \) one obtains the Rabi oscillations (see figure 2.2). If \( \tau = \pi/\Omega \) the atom will be transferred into the state \( |2\rangle \), whereas for \( \tau = \pi/(2\Omega) \) the atom ends up into a linear superposition of \( |1\rangle \) and \( |2\rangle \) with equal amplitude probabilities. These two particular pulses are commonly called \( \pi \) pulse and \( \pi/2 \) pulse.

Let us now consider an atom moving in the \( x \) direction and a light beam interacting with it at time \( t=0 \) in \( x=0 \). If the beam has a wavevector \( k \) tilted by an angle \( \theta \) with respect to the \( x \) axis, according to equation (2.21) after the pulse we will have part of the atomic wavefunction in the state \( |1\rangle \) with no extra phase and travelling on the \( x \) axis and part in the state \( |2\rangle \) with an acquired phase term \((\phi_L - \pi/2)\) and directed along \((p + \hbar k_L)/(\mid p + \hbar k_L \mid)\).

In figure 2.3 the ideal cases of \( \pi/2 \) and \( \pi \) pulses applied on an atom in the state \( |1\rangle \) and in the state \( |2\rangle \) are drawn.

Note that when the atom is in the ground state \( |1\rangle \) a \( \pi \) pulse corresponds to the absorption of a photon with energy \( \hbar \omega_L \) and to the acquisition of a momentum \(+\hbar k_L\), but when the atom is initially in the state \( |2\rangle \) stimulated emission occurs so a photon with energy \( \hbar \omega_L \) is emitted copropagating with the laser light and the atomic momentum changes by \(-\hbar k_L\).

The tools described so far are enough to realize an atom interferometer (see figure 2.1). A first \( \pi/2 \) pulse splits the atomic function in two parts that independently evolve along different paths, after a time \( T \) a \( \pi \) pulse induces an internal state inversion and an opposite momentum transfer, then again after a time \( T \) a last \( \pi/2 \) pulse lets the two interferometer arms recombine.

---

**Figure 2.3:** \( \pi \) pulses applied on atoms in the state \( |1\rangle \) or \( |2\rangle \) are illustrated on the left, \( \pi/2 \) on the right.
2.2 Atom–light field interaction

The longer the interferometer time $T$, the better the performances of the interferometer, so this experimental parameter $T$ is usually chosen as large as possible. In a free-fall experiment gravity itself, though, accelerates the atoms and common apparatus dimensions fix a limit also on $T$ to below 1 s. In this case we need a two-level atomic system whose lifetime is much longer than that.

In the picture adopted up to now a single photon, matching the atomic transition frequency, is used. In real systems only the ground state of an atom has exactly an infinite lifetime, whereas all the others have a certain probability of decaying in lower energy states. The optical transitions of the D$_2$ line, for example have a lifetime of 26 ns. Among the multitude of levels in the complex atomic spectra the two levels pair is chosen in such a way that the two-level system is a good approximation, lifetimes being orders of magnitude longer than the experimental times. One possibility is the choice of a dipole forbidden transition so that spontaneous emission has a low rate, but this implies that also the absorption probability is low. Good candidates are then two hyperfine levels of the ground state. Ground state hyperfine splittings range usually from 100 MHz to 10 GHz, falling in the microwave range. $^{87}$Rb has a ground state hyperfine splitting of 6.8 GHz. Using microwaves for implementing the interferometer components is possible and is commonly adopted in atomic fountain clocks experimental schemes, in which momentum transfers are not desired. The small transferred velocity in that case is in fact of about $10^{-4}$ mm/s.

When atom interferometers are used as inertial sensors, though, as in our experiment, the sensitivity is directly proportional to the transferred momentum. So a microwave pulse is not the best choice. A Raman transition, realized with two counterpropagating light beams having a frequency difference that matches the atomic hyperfine splitting, allows to use the two hyperfine states as a good two-level system with infinite lifetime of the upper state and also to transfer twice an optical momentum, i.e. a two-photon recoil velocity of $2\nu_r = 11.8$ mm/s.

The following section is dedicated to the detailed description of the parameters related with the Raman transition in our experiment.
2.2.2 Two-photon Raman transition on a multi-level atom

Let us consider a multi-level atomic system with two main energy levels, $|a\rangle$ and $|b\rangle$ (the two hyperfine levels of rubidium ground state $5^2S_{1/2}$), and a virtual one, $|i\rangle$. The state $|i\rangle$ represents a virtual state obtained as a sum over all\(^1\) the hyperfine levels of $5^2P_{3/2}$.

Two counterpropagating light beams

$$E_{R1}(x,t) = E_{R1,0} \cos(k_{R1} \cdot x - \omega_{R1} t + \phi_{R1})$$

$$E_{R2}(x,t) = E_{R2,0} \cos(k_{R2} \cdot x - \omega_{R2} t + \phi_{R2})$$

with frequencies $\omega_{R1}$ and $\omega_{R2}$ are simultaneously sent on the atom.

As in the previous simpler case we will consider external and internal degrees of freedom coupled together. Besides the two main states $|1\rangle$ and $|2\rangle$ three other states will be taken into account. These are the states the

\(^1\)Rubidium is an alkali atom, therefore the ground level of the fine structure has only two sublevels, $F=1$ and $F=2$, nuclear spin being $I=3/2$ as reported in appendix A. For the same reason the excited level $5^2P_{3/2}$ has a four-fold hyperfine substructure with $F=0,1,2$ and 3.
atom can be transferred to if starting from one of the two main states and interacting with one of the two light fields. One would have |i0⟩ if the atom in |1⟩ absorbs a photon from \(E_{R1}\) or if the atom in |2⟩ absorbs a photon from \(E_{R2}\). Also the other two cases are possible and basically must be considered in order to correctly obtain the energy levels light shift

\[
|1⟩ = |a, p⟩ \quad E_1 = \hbar \omega_a + \frac{p^2}{2m} = \hbar \omega_1 \\
|2⟩ = |b, p + hkr_1 - hkr_2⟩ \quad E_2 = \hbar \omega_b + \frac{[p + hkr_1 - hkr_2]^2}{2m} = \hbar \omega_2 \\
|i0⟩ = |i, p + hkr_1⟩ \quad E_{i0} = \hbar \omega_i + \frac{p^2}{2m} = \hbar \omega_{i0} \\
|i1⟩ = |i, p + 2hkr_1 - hkr_2⟩ \quad E_{i1} = \hbar \omega_i + \frac{[p + 2hkr_1 - hkr_2]^2}{2m} = \hbar \omega_{i1} \\
|i2⟩ = |i, p + hkr_2⟩ \quad E_{i2} = \hbar \omega_i + \frac{p^2}{2m} = \hbar \omega_{i2}
\]

The Hamiltonian for the described system in the electric dipole approximation is

\[
\mathcal{H} = \sum_s \hbar \omega_s |s⟩⟨s| - \mathbf{d} \cdot (E_{R1} + E_{R2}) 
\]  

(2.24)

with \(s=1,2,i0,i1,i2\). It will be applied on the atomic wavefunction

\[
|\psi(t)⟩ = \sum_s c_s(t) e^{-i\omega_s t}|s⟩.
\]  

(2.25)

As also illustrated in figure 2.4 the two light fields have a frequency difference \(\omega_{R1} - \omega_{R2}\) that can be slightly different from the atomic resonance \(\omega_0\). From now on this detuning will be called \(\delta_R\). Furthermore \(\Delta_R\) will indicate the common detuning from the excited level |i0⟩ whereas \(\Delta_1\) and \(\Delta_2\) will be respectively the detuning of the light field \(E_{R1}\) from the transition |2⟩ → |i0⟩ and the detuning of \(E_{R2}\) from |1⟩ → |i0⟩.

Defining the Rabi frequency induced by the laser \(l\) \((l=1,2)\) between the states |m⟩ \((m=1,2)\) and |n⟩ \((n=i0,i1,i2)\) as

\[
\Omega_{ml} = \frac{\langle n| - \mathbf{d} \cdot E_{R1,0}|m⟩}{\hbar}
\]  

(2.26)

we obtain the following differential equations for the coefficients in the ro-
tering wave approximation

\[ i\dot{c}_1(t) = c_0(t) \frac{\Omega_{101}^*}{2} e^{i\Delta_{R1} (t)} + c_2(t) \frac{\Omega_{122}^*}{2} e^{i\Delta_{2} (t)} \]

\[ i\dot{c}_2(t) = c_0(t) \frac{\Omega_{202}^*}{2} e^{i\Delta_{R1} (t)} + c_1(t) \frac{\Omega_{211}^*}{2} e^{i\Delta_{2} (t)} \]

(2.27)

\[ i\dot{c}_0(t) = c_1(t) \frac{\Omega_{101}}{2} e^{-i\Delta_{R1} (t)} + c_2(t) \frac{\Omega_{202}}{2} e^{-i\Delta_{2} (t)} \]

\[ i\dot{c}_1(t) = c_2(t) \frac{\Omega_{211}}{2} e^{-i\Delta_{2} (t)} \]

\[ i\dot{c}_2(t) = c_1(t) \frac{\Omega_{122}}{2} e^{-i\Delta_{2} (t)} \]

The three excited levels can be adiabatically eliminated [36] by integrating the three equations and taking out of the integrals the slowly varying terms. What we obtain is

\[
 i\dot{c}_1(t) = c_1(t) \left[ \frac{|\Omega_{101}|^2}{2\Delta_R} + \frac{|\Omega_{122}|^2}{2\Delta_2} \right] + c_2(t) \left[ \frac{\Omega_{101}^* \Omega_{202}}{4(\Delta_R - \delta_R)} e^{-i(\phi_{R1} - \phi_{R2}) + i\delta_{R1}} \right] 
\]

\[ i\dot{c}_2(t) = c_1(t) \left[ \frac{\Omega_{211}^* \Omega_{202}}{4\Delta_R} e^{i(\phi_{R1} - \phi_{R2}) - i\delta_{R1}} \right] + c_2(t) \left[ \frac{|\Omega_{202}|^2}{4(\Delta_R - \delta_R)} + \frac{|\Omega_{211}|^2}{4\Delta_1} \right]. \]

Let us now simplify the equations by defining the frequency light shifts of the two states, their sum and difference

\[ \Omega_1^{\text{AC}} = \left[ \frac{|\Omega_{101}|^2}{2\Delta_R} + \frac{|\Omega_{122}|^2}{2\Delta_2} \right] \quad \Omega_2^{\text{AC}} = \left[ \frac{|\Omega_{202}|^2}{4(\Delta_R - \delta_R)} + \frac{|\Omega_{211}|^2}{4\Delta_1} \right] \]

(2.28)

\[ \Omega^{\text{AC}} = \Omega_1^{\text{AC}} + \Omega_2^{\text{AC}} \quad \delta^{\text{AC}} = \Omega_2^{\text{AC}} - \Omega_1^{\text{AC}}. \]

The resulting equations are formally similar to those obtained in section (2.2.1) for a two-level atom system if one introduces the following effective quantities

\[ \omega_{\text{eff}} = \omega_{R1} - \omega_{R2} \]  

(2.30)

\[ k_{\text{eff}} = k_{R1} - k_{R2} = \frac{k_{R1}}{|k_{R1}|} (|k_{R1}| + |k_{R2}|) \]  

(2.31)

\[ \phi_{\text{eff}} = \phi_{R1} - \phi_{R2} \]  

(2.32)

\[ \Omega_{\text{eff}} = \frac{\Omega_{101} \Omega_{202}}{2\Delta_R} \]  

(2.33)

\[ \Omega_{\text{eff}}' = \sqrt{\Omega_{\text{eff}}^2 + (\delta - \delta^{\text{AC}})^2}. \]  

(2.34)
2.2 Atom–light field interaction

Considering that $\delta_R \ll \Delta_R$ one then has

\[
\dot{c}_1(t) = c_1(t)\Omega_{AC}^* + c_2(t)\frac{\Omega_{\text{eff}}^*}{2} e^{-i\phi_{\text{eff}} + i\delta_R t},
\]

\[
\dot{c}_2(t) = c_1(t)\frac{\Omega_{\text{eff}}^*}{2} e^{i\phi_{\text{eff}} - i\delta_R t} + c_2(t)\Omega_{AC}^*.
\]

The rotation of the coefficients

\[
c_1(t) = s_1(t)e^{i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}}, \quad c_2(t) = s_2(t)e^{-i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}},
\]

allows to remove the time dependency in the off-diagonal terms and in the end to achieve the following form for the equations

\[
\dot{s}_1(t) = \frac{1}{2} \left[ s_1(t)(\delta_R - \delta_{AC}) + s_2(t)\Omega_{\text{eff}} e^{-i\phi_{\text{eff}}} \right],
\]

\[
\dot{s}_2(t) = \frac{1}{2} \left[ s_1(t)\Omega_{\text{eff}} e^{i\phi_{\text{eff}}} - s_2(t)(\delta_R - \delta_{AC}) \right].
\]

This system can be easily solved and the general expression for the original coefficients of the atomic wavefunction is then obtained

\[
c_1(t) = e^{i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}} \left\{ c_1(0) \left[ \cos \left( \frac{\Omega_{\text{eff}} t}{2} \right) - i\frac{\Omega_{\text{eff}}}{\Omega_{\text{eff}}^*} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right] + c_2(0) e^{-i\phi_{\text{eff}}} \left[ -i\frac{\Omega_{\text{eff}}}{\Omega_{\text{eff}}^*} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right] \right\},
\]

\[
c_2(t) = e^{-i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}} \left\{ c_1(0) e^{i\phi_{\text{eff}}} \left[ -i\frac{\Omega_{\text{eff}}}{\Omega_{\text{eff}}^*} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right] + c_2(0) \left[ \cos \left( \frac{\Omega_{\text{eff}} t}{2} \right) + i\frac{\Omega_{\text{eff}}}{\Omega_{\text{eff}}^*} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right] \right\}.
\]

Below the light parameters in the two analyzed pictures are reported and compared

<table>
<thead>
<tr>
<th>two–level atom</th>
<th>(\omega_L)</th>
<th>(k_L)</th>
<th>(\phi_L)</th>
<th>(\Delta)</th>
<th>(\Omega)</th>
<th>(\Omega')</th>
</tr>
</thead>
<tbody>
<tr>
<td>single photon transition</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>multi–level atom</td>
<td>(\omega_{\text{eff}})</td>
<td>(k_{\text{eff}})</td>
<td>(\phi_{\text{eff}})</td>
<td>(\delta_R)</td>
<td>(\Omega_{\text{eff}})</td>
<td>(\Omega'_{\text{eff}})</td>
</tr>
<tr>
<td>Raman transition</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Experimentally it is possible to have $\delta_R, \delta_{AC} \ll \Omega_{\text{eff}}$ so that equations (2.36) and (2.37) are simplified in

\[
c_1(t) = e^{i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}} \left\{ c_1(0) \cos \left( \frac{\Omega_{\text{eff}} t}{2} \right) - ic_2(0) e^{-i\phi_{\text{eff}}} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right\},
\]

\[
c_2(t) = e^{-i\frac{\delta_R t}{2} - i\frac{\Omega_{AC}^* t}{2}} \left\{ c_2(0) \cos \left( \frac{\Omega_{\text{eff}} t}{2} \right) - ic_1(0) e^{i\phi_{\text{eff}}} \sin \left( \frac{\Omega_{\text{eff}} t}{2} \right) \right\}.
\]
For each of the four possible transitions induced with a pulse of temporal length $\tau$ we then have the following momentum transfers and accumulated phase shifts:

<table>
<thead>
<tr>
<th>internal state</th>
<th>momentum</th>
<th>accumulated phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>a\rangle \rightarrow</td>
<td>a\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>a\rangle \rightarrow</td>
<td>b\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>b\rangle \rightarrow</td>
<td>a\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>b\rangle \rightarrow</td>
<td>b\rangle$</td>
</tr>
</tbody>
</table>

2.2.3 Mach–Zehnder Raman interferometer

A complete interferometer sequence of Raman pulses is now studied in order to determine the phase shift imprinted in the atomic wavefunction after the Raman interferometer. As illustrated in figure 2.5, the three pulses have a temporal length $\tau - 2\tau - \tau$ and are separated by a time $T - \tau$ so that the whole sequence lasts $2T$. It could be also described as two identical Ramsey sequences ($\tau - \tau$) one after the other, that enable a coherent interference of atoms with different resonant frequencies.

Let us consider an atom initially at rest in the state $|1\rangle$. In this treatment $\phi_{\text{eff}}$ was considered the phase of the radiation at the beginning of the atom–laser interaction, so we have different $\phi_{\text{eff}}$ contributions for each pulse. Let us consider the other laser parameters constant and deduce the accumulated phase on the two possible paths by detecting the atom in the output channel 1 of the interferometer (see figure 2.6)
2.2 Atom–light field interaction

![Diagram of a three-pulse Mach–Zehnder type interferometer]

Figure 2.5: Time sequence of a three-pulse Mach–Zehnder type interferometer. The interferometer starts at $t = 0$ with a first $\frac{\pi}{2}$-pulse of length $\tau$. Then the two oscillators, the atom and the Raman field, independently evolve for a time $T - 2\tau$. A second pulse, a $\pi$-pulse of temporal length $2\tau$ is sent. Finally, again after a free evolution interval $T - 2\tau$, the last $\frac{\pi}{2}$-pulse of length $\tau$ is sent.

Path I \hspace{1cm} (1→1→2→1)

\[
\phi_I = \left(\frac{\delta R \tau}{2} - \frac{\Omega_{AC} \tau}{2}\right) + \left(\phi^{T-\tau}_{\text{eff}} - \delta_{R \tau} - \Omega_{AC} \tau - \frac{\pi}{2}\right) + \left(\frac{\delta R \tau}{2} - \frac{\Omega_{AC} \tau}{2} - \phi^{2T-\tau}_{\text{eff}} - \frac{\pi}{2}\right) = \\
\phi^{T-\tau}_{\text{eff}} - \phi^{2T-\tau}_{\text{eff}} - \pi - 2\Omega_{AC} \tau
\]

Path II \hspace{1cm} (1→2→1→1)

\[
\phi_{II} = \left(\phi^{0}_{\text{eff}} - \frac{\delta R \tau}{2} - \frac{\Omega_{AC} \tau}{2} - \frac{\pi}{2}\right) + \left(\delta_{R \tau} - \Omega_{AC} \tau - \phi^{T-\tau}_{\text{eff}} - \frac{\pi}{2}\right) + \left(\frac{\delta R \tau}{2} - \frac{\Omega_{AC} \tau}{2}\right) = \\
\delta_{R \tau} + \phi^{0}_{\text{eff}} - \phi^{T-\tau}_{\text{eff}} - \pi - 2\Omega_{AC} \tau.
\]

The acquired phase difference is then

\[
\Delta \phi = \phi_{II} - \phi_{I} = \left[\phi^{2T-\tau}_{\text{eff}} - 2\phi^{T-\tau}_{\text{eff}} + \phi^{0}_{\text{eff}}\right] + \delta_{R \tau} = \phi^{0}_{\text{eff}} - 2\phi^{T}_{\text{eff}} + \phi^{2T}_{\text{eff}}. \quad (2.40)
\]

\(\Delta \phi\) can be varied by any accelerated motion of the atom relative to the wavefronts of the Raman beams or by any relative variation between the atomic frequency and the Raman frequency difference.

The high sensitivity of the atom interferometer requires then a precise control of the several sources of phase shifts, but it also allows to measure the effect of one of those sources with a high resolution, once the others are well under control.
2.3 Interferometer phase shifts

The total phase shift that can be observed in the output of an atom interferometer can be decomposed into three contributions:

$$\Delta \phi_{\text{tot}} = \Delta \phi_{\text{laser}} + \Delta \phi_{\text{sep}} + \Delta \phi_{\text{evol}}$$

(2.41)

$\Delta \phi_{\text{laser}}$ contains the phase terms that are imprinted in the atomic wavefunction during the four space-time points in which the atom interacts with the interferometer light pulses. It is the one obtained in the previous section (equation 2.40) by solving the Schrödinger equation of the interaction between atom and Raman laser field. For a complete interferometer sequence, $x^A, x^B, x^C$ and $x^A, x^B, x^C$ being the positions where the atom is at the times $0, T, 2T$ along the two interferometer paths, it results

$$\Delta \phi_{\text{laser}} = \phi(x^A, 0) - \phi(x^B, T) - \phi(x^B, T) + \phi(x^C, 2T).$$

(2.42)

$\Delta \phi_{\text{sep}}$ arises in case of non-uniform fields. Only with uniform acceleration fields, in fact, the two wavepackets perfectly overlap at the end of the interferometer. Interference occurs anyway though, thanks to the extended spatial dimensions of the atomic wavefunction. The partial mismatching gives rise to the additional term

$$\Delta \phi_{\text{sep}} = \frac{p(x^C - x^C_H)}{2\hbar}$$

(2.43)

where $x^C$ and $x^C_H$ are the end points for the two different paths.

$\Delta \phi_{\text{evol}}$ is due to the differential free evolution of the wavepackets between the Raman pulses. Using Feynman path integrals formalism, valid
for Lagrangians at most quadratic with respect to $x$ and $v$, it can be demonstrated \cite{57} that the evolution of the wavefunction of a quantum system in an external potential $V(x)$, from $(x_A, t_i)$ to $(x_B, t_f)$ is determined by the quantum propagator $\sum_{\Gamma} e^{iS_\Gamma/h}$, where $\Gamma$ are all the possible paths and $S_\Gamma$ is the action along the path $\Gamma$. In the classical limit the action

$$S_\Gamma = \int_{t_i}^{t_f} L[x(t), \dot{x}(t)] \, dt$$

(2.44)

is much greater than $\hbar$. In this case only the paths close to the classical ones interfere constructively and the resulting phase shift is simply given by the differential action evaluated along the two possible paths

$$\Delta\phi_{\text{evol}} = \frac{S_{t_f} - S_{t_i}}{\hbar} = \frac{1}{\hbar} \int_{\Gamma_0} L \, dt$$

(2.45)

where $\Gamma_0$ indicates the whole classical interferometer path.

A few examples of phase shifts of different nature that are present in atom interferometers will be now introduced and estimated for our specific case.

From now on we will consider the limit of short pulses $\tau \ll T$. The theory of finite length pulses is described in detail in \cite{58}.

### 2.3.1 Inertial shifts

Any external force induces an acceleration on the atomic motion and, consequently, a displacement of the atom relative to the laser wavefronts. The most important inertial shifts are evaluated in this section.

**Gravity (freely falling atom)**

We will now study a vertical Raman interferometer on an atom that is freely falling under the effect of gravity. Let us consider first the gravity acceleration $g$ as constant and uniform in the interferometer region. This approximation implies that $\Delta\phi_{\text{evol}}$ and $\Delta\phi_{\text{sep}}$ vanish. The atomic vertical velocity changes linearly with time $v_z(t) = v_z(0) - gt$, therefore, in order to maintain always the resonance, or at least to keep $\delta_R$ constant in the atomic frame, the Raman lasers frequency difference in the lab frame needs to be linearly ramped. In this way the Doppler effect is kept constant.
If the starting conditions are \( z(0) = 0 \) and \( v(0) = v_0 \) we have

\[
\begin{align*}
\phi(x^A, 0) &= 0 \\
\phi(x^B, T) &= k_{\text{eff}} \left[ -\frac{1}{2} gT^2 + v_0 T \right] \\
\phi(x^B, T) &= k_{\text{eff}} \left[ -\frac{1}{2} gT^2 + \left( v_0 + \frac{h k_{\text{eff}}}{m} \right) T \right] \\
\phi(x^C, 2T) &= k_{\text{eff}} \left[ -2 gT^2 + \left( 2v_0 + \frac{h k_{\text{eff}}}{m} \right) T \right]
\end{align*}
\]

resulting in an interferometer phase shift induced by local gravity \( g \) of

\[
\Delta \phi_g = -k_{\text{eff}} g T^2. \tag{2.46}
\]

A more precise calculation taking into account the evolution of the wavepacket during the finite length pulses of the interferometer has been done in [58] obtaining

\[
\Delta \phi_g = -k_{\text{eff}} g \left( T + \frac{4\tau}{\pi} \right) (T + 2\tau). \tag{2.47}
\]

In common atom gravimeters the acceleration is not directly determined from the measured phase shift, but determining the frequency ramp that exactly compensates the Doppler shift induced by gravity during the free fall.

**Acceleration gradient**

During the interferometer sequence the two parts of the atomic wavepacket reach a maximum separation of \( \delta z = v_r T \) at the time \( T \) of the central pulse. For \( v_r = 6 \text{ mm/s} \) and \( T = 150 \text{ ms} \), as in our case, \( \delta z = 0.9 \text{ mm} \). The differential acceleration for such a distance can be detected. We can express gravity considering a reference value \( -g \) for the acceleration in \( z = 0 \) and a constant gradient \( \gamma \) (\( \sim 3 \times 10^{-6} \text{ s}^{-2} \)). Any freely falling body will then have the following equation of motion \( \ddot{z}(t) = -g + \gamma z(t) + z(0) \).

With this linear gradient approximation, the path integral approach can be applied considering an unperturbed system with uniform gravity and a perturbation to the Lagrangian \( \Delta \mathcal{L} = -\frac{1}{2} m \gamma z^2 \). The calculation leads [59] to a further shift induced by the gravity gradient that adds up to the uniform gravity one. This perturbation phase term is expressed by

\[
\Delta \phi_{\text{grad}} = k_{\text{eff}} \gamma T^2 \left( \frac{7}{12} g T^2 - v_0 T - z_0 \right). \tag{2.48}
\]
2.3 Interferometer phase shifts

The term introduced by the gradient is \( \sim -\frac{5}{12}k_{\text{eff}}g T^4 \). For our experimental values it is \( \sim 3 \times 10^{-8} \) times the dominant one. For much larger interferometer times \( T \) the sensitivity to the gradient would increase quadratically because of the larger wavepacket separation, but apparatus dimensions become a problem. Other multi-pulses interferometer schemes can be implemented for direct gravity–gradient measurements \([42]\), but the most efficient and convenient alternative for measuring gravity gradients with atoms is to use two vertically separated atomic samples (at a distance \( \Delta h \)) and simultaneously operate a Raman interferometer on them.

In this way the huge shift connected to \( g \) is cancelled out being common for the two interferometers and the measured phase difference becomes

\[
\Delta \phi_{g,\text{up}} - \Delta \phi_{g,\text{dw}} = -k_{\text{eff}}(g_{\text{up}} - g_{\text{dw}})T^2 = -k_{\text{eff}}\Delta h T^2
\]

where \( g_{\text{up}} \) and \( g_{\text{dw}} \) are the acceleration values at the points of the two atomic samples. In the same experimental conditions as before the induced phase shift is \( \Delta h \times 1.12 \text{ rad/m} \). In our experiment \( \Delta h \) is 30 cm, therefore the differential acceleration is \( \sim 10^{-7} g \).

Rotations

Exactly the same procedure adopted for the linear gradient example can be used for small rotations as well. If \( \Omega \) is the vectorial angular velocity the perturbation term is \( \Delta \mathcal{L} = m \Omega \cdot (\mathbf{x} \times \mathbf{v}) \) and the phase shift is

\[
\Delta \phi_{\text{rot}} = -2\Omega \cdot (\mathbf{v} \times \mathbf{k}_{\text{eff}})T^2.
\]  

(2.50)

Atoms in an atomic fountain are approximately launched along the vertical direction, but the finite temperature of the ensemble implies a horizontal velocity spread. Furthermore, due to launch errors, \( \mathbf{v} \) can have a small component perpendicular to \( \mathbf{k}_{\text{eff}} \). Suppose now that the Raman beams propagate exactly along the vertical direction. If the interferometer measurement is performed at a latitude \( \theta_l \), Earth rotation around its axis gives a contribution to the phase shift of

\[
\Delta \phi_{\text{rot}} = -2\Omega v_{\text{EW}} k_{\text{eff}} T^2 \cos \theta_l
\]

(2.51)

where \( v_{\text{EW}} \) is the horizontal velocity along the east–west direction. Here the horizontal velocity along north–south direction gives no shift being in the plane defined by \( k_{\text{eff}} \) and \( \Omega \).
If two atomic samples are launched in the same direction, tilted of a small angle \( \alpha \) from the vertical, up to different heights \( h_{up} \) and \( h_{dw} \), their horizontal velocity would be different so a differential Coriolis term would rise

\[
\Delta \phi_{rot,up} - \Delta \phi_{rot, dw} = -2\Omega \Delta v_{E_H} k_{\text{eff}} T^2 \cos \theta_1.
\]  

Our experiment is realized in Sesto Fiorentino (near Florence, Italy), that is located at a latitude \( \theta_1 = 43^\circ50'07'' N \). Considering that a complete 2\( \pi \) Earth rotation takes 23 h 56 min 4 s, \( \Omega = 7.29 \times 10^{-5} \) rad/s. For \( T = 150 \) ms, \( k_{\text{eff}} = 1.61 \times 10^7 \) m\(^{-1} \) and two atomic clouds launched up to \( h_{dw} = 60 \) cm and \( h_{dw} = 95 \) cm, we have a differential shift of about 7.6 mrad for each mm/s of east–west velocity of the upper cloud (corresponding to \( \alpha \sim 230 \) \( \mu \) rad). This problem could be reduced for example using a different interferometer configuration by dropping atoms instead of launching them upwards. In this case the horizontal velocity transferred would be orders of magnitude smaller. Another possibility could be the realization of an atom interferometer in an optical lattice where atoms are optically trapped.

**Magnetic fields**

Atoms with a magnetic dipole moment \( \mu \) in a magnetic field \( \mathbf{B} \) have an energy \(-\mu \cdot \mathbf{B}\) and then are subject to an acceleration

\[
a = \frac{1}{m} \nabla (\mu \cdot \mathbf{B}) = \frac{1}{m} m_{F} \mu_{F} g_{F} \frac{dB_{z}}{dz} \frac{z}{|z|}.
\]  

if the quantization axis is chosen along the \( z \) axis with a bias magnetic field. A uniform field does not accelerate the atoms, but as soon as a field spatial variation is introduced the atoms are accelerated depending on their \( m_{F} \) quantum number and on their mass \( m \).

The first non–vanishing term for the energy of atoms with \( m_{F} = 0 \) is \(-\mu_{F} g_{F} |\mathbf{B}|^2/(4\Delta E_{HF}) = \hbar a_{z,11} |\mathbf{B}|^2 / 2 \). This accelerates the atom by

\[
a = \frac{1}{2m} \nabla (\hbar a_{z,11} |\mathbf{B}|^2) = \frac{\hbar}{m} a_{z,11} B \frac{dB_{z}}{dz} \frac{z}{|z|}.
\]  

By considering a reasonable field and its gradient in the interferometer region of \( B(z) = 300 \) mG and \( B'(z) = 5 \times 10^{-5} \) G/mm we can estimate the acceleration induced in both cases of atoms in \( m_{F} = 1 \) and \( m_{F} = 0 \). For atoms
in \( m_x = 1 \) we use (2.53) and get an acceleration of \( \sim 2 \times 10^{-5} \) \( g \), whereas for \( m_x = 0 \) we use (2.54) and get \( \sim 4 \times 10^{-9} \) \( g \).

In the experiment \(^{87}\text{Rb} \) atoms in the \( m_x = 0 \) will be used, to suppress magnetically induced accelerations.

For magnetic fields inducing a uniform acceleration field or one with a constant gradient the measured phase shift with the interferometer has the same expressions as that reported for gravitational fields.

**Electric fields**

Homogeneous electric fields \( \mathbf{E} \) do not accelerate neutral particles, but a charged particle is accelerated by

\[
a = \frac{q \mathbf{E}}{m}
\]

(2.55)

where \( q \) is the charge of the particle with mass \( m \). Up to now matter neutrality has been tested down to \( \Delta(\delta q)/(\delta q) < 10^{-22} \), so it has been proposed [60] to set up atom interferometry experiments and explore possible charge imbalances between proton and electron. An electric field of \( MV/m \) allows a sensitivity on possible charge imbalances of \( \Delta(\delta q)/(\delta q) = 10^{-25} \).

Inhomogeneous electric fields can induce effects on the interferometer phase because the ground state level is shifted by \( \Delta E = -2\pi\epsilon_0\alpha E^2 \), where \( \alpha \) is the atomic electric polarizability, and an acceleration

\[
a = \frac{4\pi\epsilon_0\alpha E}{m} \frac{dE_z}{dz} \frac{z}{|z|}
\]

(2.56)

results. Based on that, experiments were realized to measure sodium [29], helium and lithium [30] electric polarizability.

### 2.3.2 Atomic frequency shifts

As described in section 2.2.2, after the splitting pulse of the interferometer a dipole moment is created in the atom. During the two \( T \) intervals of free evolution in the interferometer sequence, the atomic dipole oscillates at its proper frequency \( \omega_0 \) and so does the reference Raman field (\( \omega_{\text{eff}} \)) that induced the electric dipole. If the two frequencies are exactly matched the two oscillators will be in phase during the whole interferometer sequence, but if that is not the case a phase shift is accumulated. If the detuning \( \delta R \) remains constant though, the acquired phase shift is cancelled out being
common for the two paths. For the same reason if one of the two frequencies changes relative to the other, one gets the phase shift

$$\Delta \phi = \int_0^T \delta_R(t') \, dt' - \int_T^{2T} \delta_R(t') \, dt'. \quad (2.57)$$

**Light shift**

A light beam of intensity $I_L$ detuned by $\Delta L$ from an atomic transition induces an opposite shift on the energy levels of the transition of $\Delta E_{AC} = \pm h\Omega_L'/2$, $\Omega_L'$ being the generalized Rabi frequency defined in equation (2.15). With a negative detuning the upper level increases its energy, the opposite for a positive detuning. If the interaction time between the light and the atom is the same during the two parts of the interferometer the phase shift on the two paths is compensated, as it is for the Raman light beams themselves in equation (2.40).

If the light is turned on only during one of the two interferometer parts the measured phase shift is

$$\Delta \phi_{AC} = \int_{i_1}^{i_2} \Omega_L \, dt. \quad (2.58)$$

$\delta_{AC}$ is the differential light shift of the two hyperfine states and depends on the intensities and on the detunings of the two Raman beams illuminating the atom. It vanishes when $\Omega_2^{AC} = \Omega_1^{AC}$. Recalling (2.28) and neglecting the difference between different $|i\rangle$ states we have

$$\begin{align*}
\Omega_{1i1} & \sim \Omega_{2i1} \sim \frac{1}{3} \left( J = \frac{1}{2} \right) |e| \langle J' = \frac{3}{2} | \mathbf{E}_{R1,0} |^2 \\
\Omega_{1i2} & \sim \Omega_{2i2} \sim \frac{1}{3} \left( J = \frac{1}{2} \right) |e| \langle J' = \frac{3}{2} | \mathbf{E}_{R2,0} |^2 \\
\frac{|\mathbf{E}_{R2,0}|^2}{\Delta_R - \delta_R} + \frac{|\mathbf{E}_{R1,0}|^2}{\Delta_1} & = \frac{|\mathbf{E}_{R1,0}|^2}{\Delta_R} + \frac{|\mathbf{E}_{R2,0}|^2}{\Delta_2}.
\end{align*}$$

Since $\Delta_1 = \Delta_R + \omega_0$ and $\Delta_2 = \Delta_R - \omega_0 - \delta_R$ and taking into account that the equation can be solved only if $-\omega_0 < \Delta_R < \omega_0$, the direct absorption probability of photons from one of the two Raman beams is minimum for $\Delta_R = -\omega_0/2$, thus the Raman beams intensities $I_{R1}$ and $I_{R2}$ must obey the following condition for having a zero differential light shift

$$\frac{I_{R2}}{I_{R1}} = \frac{\Delta_1 - \Delta_R}{\Delta_R \Delta_1} \cdot \frac{\Delta_2 (\Delta_R - \delta_R)}{\Delta_2 - \Delta_R + \Delta_1} \sim -\frac{\Delta_R - \omega_0}{\Delta_R + \omega_0} = 3. \quad (2.59)$$
In the particular case of $^{87}\text{Rb}$ $\omega_0 \sim 2\pi \times 6.8$ GHz. For $\Delta R = -2\pi \times 3.4$ GHz the ratio $I_{R2}/I_{R1}$ varies of about 0.01% for any MHz variation of $\Delta R$.

**Zeeman shift**

A magnetic field $\mathbf{B}$ removes the degeneracy among $m_p$ energy sublevels. The first order Zeeman shift is

$$\Delta E_{Z,1} = \mu_B g_J m_p |\mathbf{B}|$$

that is zero for atoms in the $m_p = 0$ sublevel. For those atoms the first non zero term is the effect to the second order in the field

$$\Delta E_{Z,II} = \frac{\mu_B g_J |\mathbf{B}|}{2A} = 2a_{Z,II} |\mathbf{B}|^2$$

with a positive sign for $F=2$ and a negative one for $F=1$, therefore yielding an energy shift $2\Delta E_{Z,II}$ to the hyperfine splitting between the two $m_p=0$ states.

Turning a magnetic field on, uniform in the region around the atom (hence with no forces acting on it) during the second half of the interferometer sequence the atomic resonance varies with the external magnetic field while the reference oscillator maintains the same frequency. The result is an accumulated phase shift

$$\Delta \phi_{Z,II} = 2\pi \int_{T/2}^{2T} a_{Z,II} |\mathbf{B}(t)|^2 dt.$$  

In order to reduce the sensitivity to external and uncontrolled magnetic fields the interferometer is operated on atoms in the $m_p = 0$ state. The sensitivity to magnetic fields for such atoms is expressed by $a_{Z,II} = 575$ Hz/G². A rectangular 10 ms long magnetic pulse of 100 mG, for example, provides a phase shift of 361 mrad.

**DC Stark shift**

The DC Stark effect, occurs in presence of a static electric field $\mathbf{E}$. The differential energy splitting induced by the field on the rubidium hyperfine $m_p=0$ sublevels is given by

$$\Delta E_{DC} = h a_{DC} |\mathbf{E}|^2$$
with $a_{DC} = -3.97 \times 10^{-10} \text{ Hz/(V/m)}^2$.

If the field is uniform in space and is turned on only during the second half of the interferometer sequence, so that there is no cancellation due to reversal of the internal states, we would have the following phase shift

$$\Delta \phi_{DC} = 2\pi \int_T^{2T} a_{DC} |E(t)|^2 dt.$$  \hspace{1cm} (2.64)

A field of 1 kV/m, turned on only from $T$ to $2T$ would induce a shift of $\sim 1$ mrad on a single $T=150$ ms interferometer.
Chapter 3

Experimental apparatus

The whole apparatus realized for the experiment is described in this chapter, but more details can be found also in [61]. The apparatus can be schematically divided into four parts: the optical system, the vacuum system, the source masses and the computer control system.

3.1 Optical system

The internal state and mechanical manipulation of the atomic clouds are controlled by tuning the frequencies and by adjusting the intensities of various laser sources in the correct temporal sequence. For this reason great effort was spent in the realization and optimization of the laser system.

Henceforth $\nu_{F \rightarrow F'}$ will label the $^{87}\text{Rb}$ transition frequency $|5^2S_{1/2,F} \rightarrow |5^2P_{3/2,F'}\rangle$.

For trapping and cooling the atoms [62], radiation of two frequencies is needed: cooling and repumping light. The cooling light is $\sim 3\Gamma$ detuned$^1$ from the closed transition $\nu_{2 \rightarrow 3}$. A $^{87}\text{Rb}$ atom in the $F=2$ ground state that absorbs this radiation has a high probability of ending up in the excited state $F'=3$, then it spontaneously decays back in the $F=2$ state because of dipole selection rules. But there is a non zero probability (about 0.1%) that it goes in the $F'=2$ state. From this excited state it can either decay into the $F=1$ or into the $F=2$ ground states. Since stimulated absorption and spontaneous emission cycling rate is of the order of $\Gamma$ all the atoms would be rapidly lost in the $F=1$ channel if no repumping occurs. A radiation, slightly detuned

$^1$ $\Gamma$ is the atomic natural linewidth and is $2\pi \cdot 6.065(9)$ MHz for $^{87}\text{Rb}$ as reported in appendix A.
from the transition $\nu_{1\rightarrow 2}$ is used for this purpose.

Then the atoms can be launched using the cooling radiation, but this requires a differential control of the upwards and downwards propagating beams frequencies.

The velocity selection and the interferometer sequence is realized by Raman pulses. Depending on the pulse length the selected velocity class can be larger or narrower with an experimentally observed limit of 1/10 of the recoil velocity [34]. The two Raman beams couple the $^{87}$Rb ground state hyperfine transition and are commonly detuned from the $D_2$ line by half the ground state hyperfine splitting $\nu_{ab}$. This means that the two Raman beams frequencies lie about $\pm$ 3.4 GHz across $\nu_{2\rightarrow 3}$.

Atomic state detection after the interferometer is realized using light slightly red-detuned from $\nu_{2\rightarrow 3}$. Repumping light is added depending on whether $F=1$ or $F=2$ is detected.

All the light beams are prepared in terms of frequency, intensity and polarization on an optical table, that is covered with a plastic box preserving the lasers from air flows and acoustic noise and suppressing fast room temperature fluctuations to below 0.01°C.

The schematic drawing reported in figure 3.1 illustrates how all the light beams for the experiment are derived. Laser frequencies are roughly referenced to rubidium spectrum reported at the bottom of the picture. The frequencies used are approximately grouped into 4 bands roughly spaced by $\nu_{ab}/2$.

In order to have stable and well-defined frequencies during any part of the experimental sequence, a diode laser is frequency locked to a $^{87}$Rb sub-Doppler transition and it is used as a frequency reference for locking all the other lasers involved in the experiment.

3.1.1 Reference light

The laser used as reference is an extended cavity New Focus Vortex 6000 having 60 mW output power at 780 nm. After a double pass through an AOM that increases its frequency by $+184.2$ MHz, an EOM oscillating at $\nu_{EOM} = 5.0$ MHz creates narrow sidebands with opposite phase across the carrier. This light is sent with linear polarization into a heated (35°C) and magnetically shielded rubidium vapor cell for optically pump the atoms. The transmitted light is retro-reflected with crossed polarization and investigates
3.1 Optical system

Figure 3.1: Schematic drawing of the laser setup. Laser REF is locked to a $^{87}\text{Rb}$ sub-Doppler transition and is used as frequency reference for the whole laser system. L1 and a tapered amplifier (T.A.) are optically injected with the reference light and their output frequency is controlled through AOMs to be used respectively for detection and blow-away of atoms in $F=2$ and for trapping, cooling and launching the atoms. REP$_M$ is frequency locked to a frequency 6.8 GHz higher than REF and is used to blow-away atoms in $F=1$ and also to inject REP’s providing repumping light. RAM$_R$ is frequency locked to a frequency 3.4 GHz lower than REF to obtain one of the Raman beams. The other one is derived from laser RAM$_S$. The frequency difference between these two lasers is phase locked to the $^{87}\text{Rb}$ hyperfine ground state transition. Laser frequencies are referenced to rubidium spectrum reported at the bottom of the picture. All the reported frequencies are expressed in MHz and the intensities (on the atoms) in mW/cm$^2$. 
the atoms. Such a probe light is detected with a photodiode and electronically demodulated with the 5.0 MHz signal driving the EOM. A dispersive signal across the rubidium sub-Doppler lines is obtained. A double loop-control for low and high frequencies serves to lock the laser frequency to the atomic resonance. The low frequency loop covers a bandwidth up to 1 kHz acting on the piezo on which the grating of the extended cavity is mounted. High frequency noise (up to 120 kHz) are corrected acting on the laser current. The laser is locked on the sub-Doppler line corresponding to \( \nu_{2\rightarrow3} \), but the actual output frequency is \( \nu_{\text{REF}} = (\nu_{2\rightarrow3} - 184.2 \text{ MHz}) \), because of the AOM between laser source and rubidium cell.

Before the AOM part of the light is picked-up for frequency locking the other lasers.

### 3.1.2 Detection light

A diode laser (SHARP GH078JA2C), labelled L1 on figure 3.1, providing 60 mW at 110 mA is optically injected with the reference light (\( \nu_{\text{REF}} \)). The side port of an optical isolator is used for the injection. A double pass into an AOM is needed for increasing the laser frequency by +183.6 MHz, i.e. \( \nu_{\text{det}} = (\nu_{2\rightarrow3} - 800 \text{ kHz}) \). The light beam is coupled into two different optical fibers going towards the apparatus, one for vertically blowing-away unwanted atoms in the F=2 state and the other one for the state detection after the interferometer.

### 3.1.3 Cooling light

The higher the intensity of the trap beams, the larger the MOT loading rate \([63, 64]\). For this reason a tapered amplifier (Toptica TA100) is used. Injected with 25 mW from the reference laser, it provides an output of 500 mW that is split into two parts travelling along independent optical paths. Each path consists of a double pass through an AOM and a fiber inco coupling. The two AOMs are driven with the same +83.4 MHz signal while trapping the atoms, whereas for the several launch steps the two driving RF signals have a frequency difference proportional to the desired launch velocity (see section 4.1.2). The actual frequency \( \nu_{\text{MOTup}} \) of the upper beams (downwards propagating) ranges from \( (\nu_{2\rightarrow3} - 17.4 \text{ MHz}) \) to \( (\nu_{2\rightarrow3} - 40.2 \text{ MHz}) \) and \( \nu_{\text{MOTdown}} \) of the lower beams (upwards propagating) ranges from \( (\nu_{2\rightarrow3} - 17.4 \text{ MHz}) \) to \( (\nu_{2\rightarrow3} - 42.6 \text{ MHz}) \).
3.1.4 Repumping light

Two diode lasers (SHARP GH0781JA2C) arranged in a master–slave configuration are used for generating repumping light.

The master laser, mounted in extended cavity is frequency locked to the repumper transition $\nu_{1-2}$ by means of an optical frequency lock. Part of this laser light is overlapped with the reference light and their beat note signal is collected on a fast photodiode (HAMAMATSU G 4176-03). The beat note is mixed with the third harmonic of a programmable synthesizer (ADF4360-1 stabilized VCO) set to 2216.6 MHz. The downconverted output signal is finally mixed with a 40 MHz reference oscillator and the output provides the feedback signal that is sent to a digital phase and frequency detector (PFD), thus frequency locking the laser. Low frequencies ($\leq 1$ kHz) are corrected acting on the piezo that holds the grating and high frequencies ($\leq 130$ kHz) are directly corrected on the diode current driver. The actual frequency of this master laser is ($\nu_{1-2} - 62.4$ MHz). Part of this light is used to inject the slave laser. The rest is shifted by $-172.2$ MHz in a double-pass AOM. Resulting light has a frequency close to the $\nu_{1-0}$ ($\nu_{1-0} - 5.5$ MHz) and is used to blow-away part of the unwanted freely falling atoms in the F=1 state just after the launch.

The output beam of the slave laser is directly sent to an AOM increasing its frequency by $+68.0$ MHz. A frequency of ($\nu_{1-2} + 5.6$ MHz) results and is used as repumper for trapping and detection.

3.1.5 Raman lasers

The Raman beams wavefronts serve as a reference for the position measurement of the atoms during their free fall. Therefore their frequency difference needs to be robustly phase locked to a stable oscillator. With less stringent requirements, a frequency lock of one of the two lasers to a stable reference is needed as well. All this is realized in our experiment with a pair of diode lasers in master–slave configuration. The master is a NEW FOCUS VORTEX 6000 and the slave laser is an anti-reflection coated SACHER sal-0780-040 in extended cavity.

Master Raman frequency lock

For frequency locking, part of the master Raman light is overlapped with the reference light and a fast photodiode (NEW FOCUS 1580) detects
their beat note that is downconverted by mixing it with the third harmonic of a 1.080 GHz synthesizer (MARCONI 2024) and then by comparing the mixer output with a 10 MHz reference quartz oscillator in a digital PFD. The lock for this laser is only acting on low frequencies on the piezo. In this way the master Raman frequency becomes $\nu_{2-3} - (\nu_{ab}/2) \sim \nu_{2-3} - 3.4$ GHz.

**Raman lasers optical phase lock loop**

The slave laser light is beaten with the master's one and a New Focus 1002 fast photodiode converts the $\sim 6.8$ GHz beat note into an electronic signal that, after an amplification of 27 dB with a JCA 48-301, is again downconverted in two steps, as illustrated in figure 3.2.

A first downconversion stage is formed by a mixer (MITEQ M0408) and a fixed frequency low phase noise local oscillator (ANRITSU MG3692A) at about 6.780 GHz. The mixer intermediate frequency (IF) output is further amplified (27 dB) by a 500 MHz amplifier (MINICIRCUITS ZFL 500 LN) and finally sent to a 10 dB directional coupler. The 10% output is used for analysis and monitoring (residual phase noise measurements or observation with a spectrum analyzer) while the 90% output is sent to the RF input of a combined analog+digital PFD [65]. The LO input is provided by a 80 MHz DDS (AGILENT 33250A) used in sweep mode around 40 MHz, phase locked to the 10 MHz ANRITSU time base. During the interferometric sequence the DDS must execute phase-continuous linear frequency ramps to compensate for the gravity-induced Doppler shift.

The analog and digital detectors are mutually exclusive so that, depending on the magnitude of the phase error $|\Delta \phi|$, only one of them is active at any given time, resulting in a phase detector with both the broad capture range of digital circuits and the high speed and low noise of analog mixers.
For small $|\Delta \phi|$ only the analog phase detector (APD) generates the detector output, while the digital phase and frequency detector (DPFD) reveals the occurrence of cycle slips in the APD. The occasional cycle slip increases $|\Delta \phi|$ above a set threshold ($\pi$) so the APD is disabled and the DPFD generates the error signal until reentering the small $|\Delta \phi|$ region. The linear range of the PFD is more than $\pm 100 \pi$.

The DPFD and APD outputs drive a three paths feed-back loop acting on the slave laser injection current, both directly at the laser mount (fast current path) and via the current driver (slow current path), and on the slave laser external cavity length, via a low voltage piezo stack.

A typical spectrum of the 40 MHz closed loop beat note is shown in figure 3.3. The closed loop square modulus of the single-sided phase noise spectral density $S_\phi$ was recorded by demodulating the 40 MHz beat note at the directional coupler output.

In order to reject the DDS contribution to the phase noise, an analog mixer compares the beat note (LO input) directly to a 90° out of phase copy of the DDS signal (RF input). The Fourier transform of the IF output, scaled to rad²/Hz, is shown in figure 3.4.
In the frequency range from 5 Hz to 200 kHz $S_\phi$ can be approximated by

$$S_\phi = \begin{cases} 
\alpha_\phi / \nu & \text{if } 5 \text{ Hz} < \nu < 200 \text{ Hz} \\
\beta_\phi & \text{if } 200 \text{ Hz} < \nu < 20 \text{ kHz} \\
\gamma_\phi \nu^2 & \text{if } 20 \text{ kHz} < \nu < 200 \text{ kHz}
\end{cases}$$

with $\alpha_\phi = 3 \times 10^{-9} \text{ rad}^2$, $\beta_\phi = 1 \times 10^{-11} \text{ rad}^2/\text{Hz}$ and $\gamma_\phi = 2.5 \times 10^{-20} \text{ rad}^2/\text{Hz}^3$.

Above 200 kHz, the servo bumps of the slow and fast current loops, at 800 kHz and 3 MHz respectively, are clearly visible. In the following we will refer to the $1/\nu$, white, and $\nu^2$ noise zones of $S_\phi$ as zone I, II, and III, respectively, while the servo bumps region, above 200 kHz, will be referred to as zone IV. The additional phase noise introduced along the optical path from the lasers to the atoms was measured to be negligible. By integrating $S_\phi$ we obtain $\sqrt{\langle \phi^2 \rangle} \sim 100 \text{ mrad}$ in the 5 Hz–10 MHz bandwidth. The contribution to $\langle \phi^2 \rangle$ from each one of the four $S_\phi$ noise zones is reported in table 3.4.

To evaluate the impact of $S_\phi$ on $\Delta \phi^2_{\text{OPLL}}$ the procedure outlined in [66] can be followed. As a result, the contribution $\Delta \phi^2_{\text{OPLL}}$ of the OPLL phase noise to $\Delta \phi^2$ is given by a weighted value of the $S_\phi$ integral

$$\Delta \phi^2_{\text{OPLL}} = \int_0^\infty S_\phi(\nu)|H(\nu)|^2 d\nu.$$  \hspace{1cm} (3.1)

$H(\nu)$ (shown in figure 3.5) is the Fourier transform of the interferometer sensitivity function $h(t)$, given by

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|}
\hline
$S_\phi$ & $\langle \phi^2 \rangle$ [rad$^2$] & $\Delta \phi^2_{\text{OPLL}}$ [rad$^2$] \\
\hline
I & $1 \times 10^{-8}$ & $9 \times 10^{-8}$ \\
II & $2 \times 10^{-7}$ & $4.6 \times 10^{-7}$ \\
III & $6.7 \times 10^{-5}$ & $2.3 \times 10^{-7}$ \\
IV & $1.1 \times 10^{-2}$ & $3.4 \times 10^{-7}$ \\
\hline
total & $1.1 \times 10^{-2}$ & $1.1 \times 10^{-6}$ \\
\hline
\end{tabular}
\caption{Contributions to the mean square phase noise angle $\langle \phi^2 \rangle$ of the OPLL and to the square of the interferometer phase resolution $\Delta \phi^2_{\text{OPLL}}$ from the four noise zones of $S_\phi$.}
\end{table}
Figure 3.4: Square modulus of the single-sided phase noise spectral density $S_\phi$ from 5 Hz to 10 MHz of the optical phase lock loop. The spectrum is divided in four different regions according to the power laws that they follow.

\[
h(t) = \begin{cases} 
  \omega_0 \cos[\omega_0(t + T)] & \text{for } -T \leq t \leq -T + \tau, \\
  \omega_0 \cos[\omega_0(t - T)] & \text{for } T - \tau \leq t \leq T, \\
  -\omega_0 \cos[\omega_0 t] & \text{for } -\tau \leq t \leq \tau, \\
  0 & \text{otherwise}
\end{cases}
\]

where $2T$ and $\tau$ were defined in section 2.2.2 and $\omega_0 = 2\pi \nu_0 = \pi/(2\tau)$.

Therefore one obtains

\[
|H(\nu)|^2 = \frac{16\nu_0^4}{(\nu^2 - \nu_0^2)^2} \sin^2(\pi\nu T) \left\{ \sin [\pi\nu(T - 2\tau)] + \frac{\nu}{\nu_0} \cos(\pi\nu T) \right\}^2 .
\]

Low frequency noise $\nu < 1/(4\pi T)$ and high frequency noise $\nu > 1/(4\tau)$ are attenuated by $|H(\nu)|^2$. The total value of $\Delta\phi_{\text{OPLL}}^2$ is 1.1 mrad, with a comparable contribution from all the four zones.

**Optical scheme for Raman beams**

The light on the other output of the polarizing beam splitter is directed to the experimental setup.

A self-made tapered amplifier (EAGLEYARD EYP-TPA-0785-00000-3006 chip) is injected with approximately 10 mW power per Raman beam and generates a total power of about 500 mW (operating current: 2.1 A). This
amplification \( \times 25 \) is necessary since power losses along the optical path to the vacuum system are about 80\% and a total power of 100 mW is needed on the atoms. Before and after the tapered amplifier two rubidium cells absorb the resonant light components in order to reduce the direct absorption probability of photons from one of the two Raman lasers. Then an AOM, basically used as fast shutter, and therefore able to generate short Raman pulses of 50–100 \( \mu \)s with resolution below 1 \( \mu \)s, shifts both frequencies by about \( +80 \text{ MHz} \) and finally the beams are coupled into a 10 m optical fiber that cleans the spatial mode while preserving the linear polarization of the incoming light. At the output of the fiber a 1000 mm focus lens collimates the Raman beams (waist: 10 mm) that are now ready to enter the vacuum system. They enter from below with equal linear polarization and exit from above. After passing through a quarter–wave plate they are retro–reflected by a mirror, thus obtaining a \( \text{lin} \perp \text{lin} \) configuration in the interferometer region. The mirror alignment is controlled by means of a tiltmeter (APPLIED GEOMECHANICS 755-1129) that has a sensitivity of about 0.1 \( \mu \)rad and a range of \( \pm 1 \) degree. The accuracy of the tiltmeter is not a fundamental parameter because of the cancellation due to the differential scheme. In the
lab frame we have in the end $\omega_{R1}$ and $\omega_{R2}$ both upwards and downwards propagating ($k_{eff} = 1.610566 \times 10^7$ m$^{-1}$). In the atomic frame, though, the Doppler effect induces opposite shifts for counter propagating light beams and the atoms, moving with velocity $v(t) = (v_0 - gt)$ see

<table>
<thead>
<tr>
<th>LAB</th>
<th>ATOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{R1}$ ↑</td>
<td>$\omega_{R1} + k_{R1}(v_0 - gt)$</td>
</tr>
<tr>
<td>$\omega_{R1}$ ↓</td>
<td>$\omega_{R1} - k_{R1}(v_0 - gt)$</td>
</tr>
<tr>
<td>$\omega_{R2}$ ↑</td>
<td>$\omega_{R2} + k_{R2}(v_0 - gt)$</td>
</tr>
<tr>
<td>$\omega_{R2}$ ↓</td>
<td>$\omega_{R2} - k_{R2}(v_0 - gt)$</td>
</tr>
</tbody>
</table>

The Doppler effect is compensated with a linear frequency sweep of Raman slave laser $\omega_{R2}(t) \rightarrow (\omega_{R2} + \gamma t)$ resulting in a perfect matching with $\omega_{R1}$ ↑ and $\omega_{R2}$ ↓ for $\gamma = -k_{eff}g$

<table>
<thead>
<tr>
<th>LAB</th>
<th>ATOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{R1}$ ↑</td>
<td>$\omega_{R1} + k_{R1}(v_0 - gt)$</td>
</tr>
<tr>
<td>$(\omega_{R2} + \gamma t)$ ↓</td>
<td>$\omega_{R2} - k_{R2}(v_0 - gt)$</td>
</tr>
</tbody>
</table>

The other beams pair has a frequency difference of $2k_{eff}gt$, so for each m/s of velocity the resonance is detuned by $\sim 2\pi \cdot 5$ MHz.

### 3.1.6 Fiber system

A fiber system is used to transfer light from the optical table to the vacuum apparatus, that was mounted on a different table. This choice presents several advantages:

- Working on the experimental apparatus is possible, leaving the lasers unperturbed and protected by external vibrations sources.
- Optical systems before and after the fibers are mechanically decoupled, so misalignment problems can be easily and quickly solved.
- At the cost of some power loss, fibers provide spatial mode cleaning.

All the fibers are polarization maintaining. They are characterized by two refractive indices, $n_1$ and $n_2$, relative to two orthogonal axis. With temperature variations $n_1/n_2$ changes and strong polarization fluctuations
in the output are possible in case the incoming polarization is not oriented along one of the two fiber axes. Once the incoming beam enters the fiber with the right polarization angle within half a degree, the output polarization is preserved and kept stable to below 0.1%.

For the cooling beams a compact system of fiber splitters is also used. There are two independent splitters for upper and lower MOT beams (see section 4.1.1). Each of them has two fiber inputs, one for the cooling beam and the other for the repumper. The cooling light is split into three outputs, whose relative power can be externally adjusted by rotating half-wave plates between internal polarizing beam splitters. Each of the three outputs is recoupled into independent fibers that are directly connected to the trap vacuum chamber. Repumping light is sent only to one output.
3.2 Vacuum system

The experiment is performed in an UHV environment (~ $10^{-9}$ mbar) in order to minimize problems related with collisions with thermal background gases. Our vacuum system, that is schematically illustrated in figure 3.7, has an elongated vertical structure due to the atomic fountain geometry. It consists of a lower trap chamber, a long vertical tube and a central chamber for detection.

3.2.1 Trap chamber

The trap chamber was made of a titanium alloy (TiAl₆V₄), that is particularly light (4430 kg/m³), hard and non magnetic. Furthermore it has a high resistivity (168 $\mu\Omega$cm²). This physical property is useful to quickly damp undesired eddy currents induced by varying electromagnetic fields.

The chamber was machined starting from a 15 cm cube by cutting all the edges orthogonally to the diagonals to obtain 8 triangular faces. In each of the 6 square faces were drilled 50 mm diameter holes to provide optical access for the trapping beams. Each triangular face has a 35 mm diameter hole.

The trapping beams are in the usual 1–1–1 configuration. For fine adjustment of the launching direction this chamber was not rigidly fixed to the rest of the apparatus, but connected with a flexible bellow. Once it is positioned in the desired orientation the cube can be fixed to the optical table independently on the rest of the vacuum system.

Figure 3.7: The vacuum system of the experiment. From the bottom, the trap chamber is connected, via a flexible coupling, to the detection chamber; the interferometer tube is on the top.
3.2.2 Interferometer tube

The atom interferometer is very sensitive to magnetic fields. For this reason it is performed in a vertical tube made of the non magnetic, low resistivity material TiAl$_6$V$_4$, as the trap chamber.

The tube is 1 m long, it has an internal diameter of 35 mm and an external one of 40 mm. These dimensions were chosen in order to be able to send through the apparatus Raman beams with a waist$^2$ of 10 mm, thus with wavefronts flat enough to be allowed to neglect the effect of their curvature on the region occupied by the atomic ensemble, and simultaneously with reduced scattering from the side walls. Furthermore we want the gravitational field source masses to be positioned as close as possible to the atoms.

A 1 m long coil is wrapped around a plastic cylinder outside the interferometer tube for generating a vertical magnetic field used to set a reference quantization axis. 10 shorter coils are wrapped one above the other around the long one in order to be able to produce localized magnetic fields. Relevant coils parameters are reported below.

<table>
<thead>
<tr>
<th>COIL</th>
<th>wire $\varnothing$</th>
<th>Radius</th>
<th>Length</th>
<th>N</th>
<th>R</th>
<th>B(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[mm]</td>
<td>[mm]</td>
<td>[mm]</td>
<td></td>
<td></td>
<td>G/A</td>
</tr>
<tr>
<td>Long</td>
<td>0.80(7)</td>
<td>32</td>
<td>926</td>
<td>1065</td>
<td>7.4</td>
<td>14.5</td>
</tr>
<tr>
<td>Short</td>
<td>0.80(7)</td>
<td>33</td>
<td>87</td>
<td>100</td>
<td>0.70</td>
<td>11.53</td>
</tr>
</tbody>
</table>

External magnetic fields are attenuated by a double-layer $\mu$-metal shield placed around the tube and its coils. The shields are two coaxial 0.76 mm thin cylinders of 1028 mm length and diameters of 74 and 95 mm respectively extending along the entire tube. Simulations showed that in the central region the axial fields are attenuated by 69 dB and radial ones by 76 dB. Fields of about 50 G saturate the $\mu$-metal shield. Direct attenuation measurements with atoms will be performed to check the simulation results.

3.2.3 Detection chamber

The detection chamber, installed between the trap chamber and the interferometer tube, has a cylindrical shape (170 mm internal diameter and 80 mm height) and 8 large portviews of 60 mm diameter on the side. These are used for detecting atoms after the interferometer, for connecting the whole

$^2$Radial distance from the beam axis at which the intensity is $1/e^2$ smaller than on the axis.
3.3 Source masses and support

vacuum apparatus to the pumping system and some are left for a further cooling stage that might be added in future.

3.2.4 Pumping system

A 751/s ion pump (VARIAN VAC Ion 75 plus Starcell) is used to keep the whole apparatus under vacuum. Approximately once every three months a titanium sublimation pump is used to lower the rubidium background pressure and clean the system.

When the surface outgassing and the pumping system reach the equilibrium the pressure level ranges from $10^{-10}$ to $10^{-9}$ mbar.

3.2.5 Atomic source

The rubidium source consists of 4 solid dispensers (SAES getters 5G0807). They are connected to an electrical feed-through so that a current control from the outside is possible. The typical operating current is $4.0 - 4.5$ A.

3.3 Source masses and support

The accuracy level on the determination of $G$ is directly related to that of the source masses position. So atoms/masses relative distances, geometrical shape of the source masses and their density distribution must be all known with high accuracy. Particular attention was paid [67] to the choice of the source masses material and to the studies of its density and geometrical properties. The robust support was carefully designed and realized as well in order to hold and position the source masses at the desired vertical height with micrometric precision.

3.3.1 Source Masses

Material physical properties

Well-defined masses with a high density are needed in an experiment that aims to measure the gravitational constant $G$. Among high density materials uranium and mercury were discarded because they are toxic. Platinum and iridium are simply too rare. Tantalum and gold are impossibly expensive. Lead is so soft that it can be easily deformed by handling. Pure tungsten is fragile at room temperature and consequently hard to machine.
Sintered tungsten materials on the other hand are machinable while maintaining high density.

Some physical properties are also desirable for the source masses: obviously a low thermal expansion coefficient and nice magnetic properties namely low magnetic susceptibility and high resistivity.

For all these reasons a particular kind of sintered tungsten (Inermet IT180) was chosen. This material was also used in the experiment of Faller et al. [68]. Its physical properties are reported in table 3.2 together with the cylinders dimensions.

<table>
<thead>
<tr>
<th>material</th>
<th>Inermet IT180</th>
</tr>
</thead>
<tbody>
<tr>
<td>composition</td>
<td>W 95.3%</td>
</tr>
<tr>
<td></td>
<td>Ni 3.2%</td>
</tr>
<tr>
<td></td>
<td>Cu 1.5%</td>
</tr>
<tr>
<td>radius</td>
<td>50 mm</td>
</tr>
<tr>
<td>height</td>
<td>150 mm</td>
</tr>
<tr>
<td>nominal density</td>
<td>18000 kg/m³</td>
</tr>
<tr>
<td>elasticity modulus</td>
<td>360 GPa</td>
</tr>
<tr>
<td>rigidity modulus</td>
<td>140 GPa</td>
</tr>
<tr>
<td>hardness</td>
<td>298 HV 10</td>
</tr>
<tr>
<td>resistivity</td>
<td>$12 \times 10^{-8}$ Ωm</td>
</tr>
<tr>
<td>thermal conductivity @ $\rho=18000$ kg/m³</td>
<td>110 W m⁻¹ K⁻¹</td>
</tr>
<tr>
<td>linear thermal expansion @ 20°C</td>
<td>$5.2 \times 10^{-6}$ K⁻¹</td>
</tr>
<tr>
<td>volume magnetic susceptibility</td>
<td>$66 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 3.2: Physical properties and dimensions of Inermet IT 180 cylinders used for the final $G$ measurement.

**Realization**

A brief description of the procedure used by the German company Plansee to realize the source masses is reported here.

Fine tungsten particles, with diameters of about 10 – 50 µm, are mixed with small percentages of nickel (3.2%) and copper (1.5%) acting as bonding materials. This mixture is placed in empty containers with the desired shape and is hydrostatically pressed. The resulting block is heated up to 1500°C in an oven, so that nickel and copper melt and fill the gaps between tungsten grains. The latter remain solid since tungsten melting point is much higher.
(3422°C). After this process the block is cooled down to room temperature and nickel and copper solidify bonding tungsten grains.

The resulting material is as machinable as the stainless steel, it does not tarnish and exhibits a particularly low thermal expansion coefficient.

Depending on the initial composition the final density can range between 17800 and 18200 kg/m³. In the same furnace run though, the density difference between two different blocks lies around 0.1% (see table 3.3).

Holes of about 150 μm diameter like the one shown in figure 3.10 have been observed with a microscope in the blocks. These holes result from the volume reduction happening during the phase transition from liquid to solid of the bonding materials. The distribution of holes within the blocks is not homogeneous because of the thermal gradients arising during the cooling process. Outer regions cool down first inducing bonding material migration from the internal to the external regions. Thus the final product has density gradients or even porous domains with no bonding materials. The imperfections distribution due to this process is not predictable and therefore they must be suppressed.

One way to reduce the number and the dimension of the holes is to machine small blocks. In fact the smaller the blocks, the more the thermal gradient is confined. And this is the reason why we use [67] many small cylinders (see Geometrical configuration) instead of a single big toroidal mass as it is done by Nolting et al. (1999) [69] and Schlamminger et al. (2002) [16].

A further step towards a more homogeneous density distribution is obtained by means of a particular treatment applied on the blocks after being sintered and called hot isostatic pressing (HIP). The combination of high temperature (1200°C) and pressure (1000 bar) compresses the blocks reducing the number and the dimensions of the holes. After this treatment the density is increased by 1% (see table 3.3).

**Geometrical configuration**

24 equal cylinders with a diameter of 100.00 mm and a height of 150.20 mm are used for the $G$ measurement. They are divided into two sets of 12 cylinders and are symmetrically arranged on two holders, described in section 3.3.2. As shown in Figure 3.8, each set has a group of six cylinders in contact with an internal ring and another group of six cylinders placed just around them. The 12 cylinders form an arrangement with a hexagonal
<table>
<thead>
<tr>
<th>Number</th>
<th>Label</th>
<th>Density before [kg/m³]</th>
<th>Density after [kg/m³]</th>
<th>Height [mm]</th>
<th>φ 1 [mm]</th>
<th>φ 2 [mm]</th>
<th>Mass [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>HIP</td>
<td>18030</td>
<td>18259</td>
<td>150.208</td>
<td>100.016</td>
<td>100.017</td>
<td>21.554</td>
</tr>
<tr>
<td>2</td>
<td>HIP</td>
<td>18052</td>
<td>18275</td>
<td>150.205</td>
<td>100.022</td>
<td>100.025</td>
<td>21.554</td>
</tr>
<tr>
<td>3</td>
<td>HIP</td>
<td>18038</td>
<td>18272</td>
<td>150.202</td>
<td>100.011</td>
<td>100.016</td>
<td>21.550</td>
</tr>
<tr>
<td>4</td>
<td>HIP</td>
<td>18018</td>
<td>18258</td>
<td>150.212</td>
<td>100.013</td>
<td>100.012</td>
<td>21.552</td>
</tr>
<tr>
<td>5</td>
<td>HIP</td>
<td>18053</td>
<td>18261</td>
<td>150.202</td>
<td>100.030</td>
<td>100.019</td>
<td>21.556</td>
</tr>
<tr>
<td>6</td>
<td>HIP</td>
<td>18063</td>
<td>18258</td>
<td>150.225</td>
<td>100.021</td>
<td>100.024</td>
<td>21.552</td>
</tr>
<tr>
<td>7</td>
<td>HIP</td>
<td>18066</td>
<td>18267</td>
<td>150.193</td>
<td>100.010</td>
<td>100.014</td>
<td>21.544</td>
</tr>
<tr>
<td>8</td>
<td>HIP</td>
<td>18076</td>
<td>18272</td>
<td>150.227</td>
<td>100.023</td>
<td>100.014</td>
<td>21.552</td>
</tr>
<tr>
<td>9</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>10</td>
<td>N/A</td>
<td>18048</td>
<td>18270</td>
<td>150.204</td>
<td>100.016</td>
<td>100.022</td>
<td>21.548</td>
</tr>
<tr>
<td>11</td>
<td>N/A</td>
<td>18043</td>
<td>18257</td>
<td>150.210</td>
<td>100.020</td>
<td>100.023</td>
<td>21.552</td>
</tr>
<tr>
<td>12</td>
<td>N/A</td>
<td>18046</td>
<td>18258</td>
<td>150.205</td>
<td>100.018</td>
<td>100.010</td>
<td>21.552</td>
</tr>
<tr>
<td>13</td>
<td>N/A</td>
<td>18001</td>
<td>18263</td>
<td>150.224</td>
<td>100.015</td>
<td>100.020</td>
<td>21.558</td>
</tr>
<tr>
<td>14</td>
<td>N/A</td>
<td>18033</td>
<td>18274</td>
<td>150.224</td>
<td>100.014</td>
<td>100.008</td>
<td>21.554</td>
</tr>
<tr>
<td>15</td>
<td>N/A</td>
<td>18052</td>
<td>18257</td>
<td>150.207</td>
<td>100.013</td>
<td>100.016</td>
<td>21.552*</td>
</tr>
<tr>
<td>16</td>
<td>N/A</td>
<td>18021</td>
<td>18259</td>
<td>150.205</td>
<td>100.012</td>
<td>100.016</td>
<td>21.552</td>
</tr>
<tr>
<td>17</td>
<td>N/A</td>
<td>18042</td>
<td>18270</td>
<td>150.202</td>
<td>100.013</td>
<td>100.016</td>
<td>21.548</td>
</tr>
<tr>
<td>18</td>
<td>N/A</td>
<td>18060</td>
<td>18269</td>
<td>150.208</td>
<td>100.016</td>
<td>100.021</td>
<td>21.548</td>
</tr>
<tr>
<td>19</td>
<td>N/A</td>
<td>18055</td>
<td>18256</td>
<td>150.200</td>
<td>100.020</td>
<td>100.021</td>
<td>21.550</td>
</tr>
<tr>
<td>20</td>
<td>N/A</td>
<td>18040</td>
<td>18274</td>
<td>150.211</td>
<td>100.027</td>
<td>100.015</td>
<td>21.552</td>
</tr>
<tr>
<td>21</td>
<td>N/A</td>
<td>18009</td>
<td>18261</td>
<td>150.202</td>
<td>100.025</td>
<td>100.017</td>
<td>21.556</td>
</tr>
<tr>
<td>22</td>
<td>N/A</td>
<td>18055</td>
<td>18270</td>
<td>150.211</td>
<td>100.002</td>
<td>100.011</td>
<td>21.548</td>
</tr>
<tr>
<td>23</td>
<td>N/A</td>
<td>18012</td>
<td>18259</td>
<td>150.193</td>
<td>100.019</td>
<td>100.027</td>
<td>21.554</td>
</tr>
<tr>
<td>24</td>
<td>N/A</td>
<td>18035</td>
<td>18272</td>
<td>150.203</td>
<td>100.013</td>
<td>100.017</td>
<td>21.550</td>
</tr>
<tr>
<td>25</td>
<td>N/A</td>
<td>18003</td>
<td>18256</td>
<td>150.157</td>
<td>100.031</td>
<td>100.014</td>
<td>21.550</td>
</tr>
<tr>
<td>26</td>
<td>N/A</td>
<td>18047</td>
<td>18275</td>
<td>150.210</td>
<td>100.028</td>
<td>100.015</td>
<td>21.554</td>
</tr>
<tr>
<td>27</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>18039</td>
<td>18265</td>
<td>150.206</td>
<td>100.018</td>
<td>100.017</td>
<td>21.5517</td>
</tr>
<tr>
<td>St. dev.</td>
<td></td>
<td>0.0022</td>
<td>0.0007</td>
<td>0.0014</td>
<td>0.0007</td>
<td>0.0005</td>
<td>0.0031</td>
</tr>
</tbody>
</table>

Table 3.3: Data relative to the 27 Inconel cylinders realized for the experiment. Pieces no. 9 and 27 were analyzed first to check their density properties. The remaining 25 were HIP treated and machined. Density levels before and after the HIP are reported (measurements performed by Plansee). The diameter, measured in two orthogonal directions around the symmetry axis, and height are indicated as well. The reported mass values refer to a direct measurement with a 1 g resolution and accuracy calibrated electronic balance OAC-24 (performed in the lab). *Cylinder number 15 was not used for the measurement.
symmetry around the central axis of the system (that is also the axis along which the atoms are launched up and freely fall down).

The adopted symmetry prevents induced horizontal acceleration along the axis. Using 24 small cylinders instead of two big tori gives few advantages: first, one can manipulate these cylinders more easily than a heavy torus; second, the density distribution is much more controllable for smaller pieces, therefore we gain in accuracy despite the increase of statistical uncertainty due to the positioning error of 24 cylinders instead of 2 tori; third, small cylinders can be rotated around their own symmetry axis and can be interchanged one with the other reducing systematic error sources related with unknown inhomogeneities.

**Masses characterization**

In an accurate measurement of $G$ an experimental signal is compared to the one obtained from simulating the experiment. The simulation contains the source masses distribution around the probe masses, therefore a bad knowledge of the source masses in terms of density distribution and geometrical shape is directly translated into a systematic error. For this reason we carried on several studies that will be described in the following.

**Ultrasonic test**

Each cylinder is scanned with ultrasounds. The ultrasonic test is performed in water where the cylinder is horizontally placed and rotated around its axis by $3^\circ$ at a time until $360^\circ$ are scanned. For a certain angle ultrasound waves are horizontally sent, propagate through the cylinder and the echo
signal from the backwall of the cylinder comes back and is detected. Every pore will reduce the echo signal as it is not solid metal and the sound is differently reflected. The resolution obtained using this technique limits the detection to imperfection larger than 1 mm. On figure 3.9 the ultrasonic test performed on one of the cylinders before and after the HIP treatment is shown. The same ultrasound power and amplification was used for recording both images and a clear density homogeneity improvement is reached. However the test was performed directly by the company (PLANSEE) that realized the cylinders and we could not obtain from them any quantitative data.

**Microscope analysis**

As already stated, the sintering process gives rise to the formation of holes with a diameter of $\sim 100 \, \mu m$ inside the material that are greatly reduced in number and dimensions with the HIP. A sample cylinder was observed with an electronic microscope before and after the HIP treatment. In Figure 3.10 an example of such holes is shown. Tungsten grains are clearly visible, surrounded by the the bonding materials.

![Microscope observation of a sintered Inconel sample, before the HIP treatment. Tungsten grains are surrounded by bonding materials (nickel and copper). A hole of about 100 $\mu m$ of diameter is clearly detected.](image)
3.3 Source masses and support

Surface studies

Only one of the test cylinders was further machined and polished. Its surface quality was analyzed with a precise measuring machine (BROWN & SHARPE SCIROCCO-DEA). Length measuring accuracy along any direction in space is according to ISO 10360-2/VDI/VDE 2617 standards.

The cylinder was fixed on a measuring stand and scans with 100 points on the side and 50 on the upper plane were made. A calibrated spherical ruby head, with a radius of 3 mm precise within 1 µm, was slowly moved towards the cylinder and brought into contact. It registered the contact position and the cylinder surface was scanned. The upper points were fitted with a plane and the side ones with a cylindric surface. The surface resulted well machined both in terms of precision and accuracy. The table below shows the declared diameter and the measured one. This procedure will be repeated on all the cylinders after the final polishing.

<table>
<thead>
<tr>
<th>declared Ø [mm]</th>
<th>measured Ø [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.849±0.001</td>
<td>99.845±0.001</td>
</tr>
</tbody>
</table>

Destructive density test

A precise knowledge of the cylinders’ density is not enough to guarantee a faithful simulation of the gravitational field that they produce. The density homogeneity level within the cylinder volume has to be somehow checked. We performed a destructive test on one of the 27 cylinders (number 27). By the Masses and Volumes section of the Italian metrological institute INRIM (formerly IMGC) high precision density measurements could be performed. Their instruments reach the highest precision for samples of (500.0±0.1) g. For this reason, trying to get as much information as possible on the density distribution, the cylinder was cut into 15 parallelepipedal shaped blocks with a squared basis (25.2×25.2) mm² and 43.9 mm high. The 15 blocks were taken out of three slices, one at the top (A), one at the center (B) and one at the bottom (C). From each slice four blocks (1,2,3,4) were taken as close as possible to the lateral surface and one (5) on the cylinder’s axis. Figure 3.11 shows the blocks positions in the original cylinder.

The density was determined by using the hydrostatic weight method, based on the Archimedes principle. It consists in determining the up thrust
force that a body wholly immersed in a fluid experiences. That thrust is equal to the weight of the displaced fluid.

The mass was measured in a standard way, in air by comparison with certified masses, taking the air thrust into account. For the volume determination a hydrostatic weighing was done, using bi-distilled water as a reference. Air density was deduced from the atmospheric pressure, air temperature and humidity values [70, 71]. Water density was determined as a function of a temperature measurement (measurement uncertainty of 0.01°C) and of an atmospheric pressure one, by using the equation recommended by CIPM (Comité international des poids et mesures) [72].

The blocks were compared in air with the standard masses in groups of three, using the double exchange weighing measurement scheme. A similar procedure was adopted for the hydrostatic measurements having the standard masses in air. Both sets of measurements have been performed using a Mettler AX1005 balance (capacity: 1100 g, sensitivity: 0.01 mg, reproducibility: 0.02 mg).

The least squares method was used to estimate mass and volume values for all the 15 blocks and the density was deduced by their ratio.

A first set of measurements gave unexpected results. A decrease in weight was observed with time. This behaviour was understood and explained con-
3.3 Source masses and support

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>18260.51(27)</td>
<td>18255.29(29)</td>
<td>18249.59(27)</td>
<td>18252.77(29)</td>
</tr>
<tr>
<td>B</td>
<td>18248.93(31)</td>
<td>18246.19(29)</td>
<td>18238.66(31)</td>
<td>18240.30(34)</td>
</tr>
<tr>
<td>C</td>
<td>18268.93(27)</td>
<td>18261.60(29)</td>
<td>18254.76(29)</td>
<td>18262.14(29)</td>
</tr>
</tbody>
</table>

Table 3.4: Density values of the 15 blocks cut out of a test Inconel718 cylinder. All values are expressed in kg/m³. The average density results 18249 kg/m³ and the standard deviation 12 kg/m³.

considering that the blocks were roughly turned and the surfaces were not precisely machined and flat. This enhanced the probability of oxide formation. This oxide is soluble in water and resulted in a mass reduction during the experiment. Up to 700 mg were lost by a single block in two weeks time (520 µg per hour) for this reason.

The set of measurements was then repeated after polishing the surfaces with a diamond paste and the observed mass loss reduced to less than 200 µg per hour of water bath.

With this measurement method the 15 blocks exhibited an average density of 18249 kg/m³ and a relative density variation of $6.6 \times 10^{-4}$ considering the standard deviation and of $2.6 \times 10^{-3}$ considering the maximum density difference. A clear density reduction (see Figure 3.11) is observed when moving towards the center both in the radial and in the axial direction. The minimum measured density value is in fact for block B5, as expected because of residual porous regions due to the thermal gradients after the sintering process, not removed by HIP treatment. Assuming that all the cylinders have a similar density distribution this effect can be easily included in the simulation.

3.3.2 Support and elevator

In our experimental scheme the source masses need to be hold and moved with high positioning precision. The mass holder and elevator was designed, in collaboration with Bruno Dulach (INFN–LNF), in order to attenuate vibrations, strong enough to hold all the cylinders without observable bending or deformations and with independent positioning control for the two sets of masses. The device was manufactured by RMP in Rome.
Figure 3.12: (Left) Picture of the mass holder and elevator. Two of the four legs are visible on the bottom; above them the reference horizontal plate and the four columns connected on the top by a circular plate. The two moving platforms are holding two sets of cylinders. (Right) Vertical cut of the platform holding the cylinders. The three materials with which it is realized are shown with different colors. The $z$ axis is the central axis along which the atoms fall.

**Technical description**

A picture of the mass holder is reported in figure 3.12. Four 3.5$'$ diameter legs hold the whole structure around the vacuum system and lie on the same air-floating optical table. The movable parts, that need to be known with particular accuracy in terms of density with respect to the elements at rest since they contribute to the gravitational signal, are two large disk-shaped platforms with a hole in the center large enough to fit the interferometer tube and the magnetic shields. The materials used for the two platforms (drawn in figure 3.12), their densities and geometrical parameters are reported in table 3.5.

Each platform is held and moved by two 480 mm long precision screws (IKO TU86S) that have a diameter of 15 mm and a pitch of 10 mm.

The movement of each screw is controlled by a step motor (Lin Engineering 5718L-03) with an angular resolution of 1.8$'$ corresponding to 1/200 of a turn, followed by a 30:1 gear (Bonfigli VF30-30F HSB6) and a protection clutch that acts whether a torque higher than 5 Nm is applied. In this way the resolution on the vertical motion is 1.7 $\mu$m.
3.3 Source masses and support

<table>
<thead>
<tr>
<th>Part</th>
<th>Material</th>
<th>Density [kg/m$^3$]</th>
<th>$R_{\text{int}}$ [mm]</th>
<th>$R_{\text{ext}}$ [mm]</th>
<th>height [mm]</th>
<th>position [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner ring</td>
<td>AISI 316L</td>
<td>7650</td>
<td>47.0</td>
<td>50.0</td>
<td>25.0</td>
<td>12.5</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>47.0</td>
<td>49.8</td>
<td>25.0</td>
<td>-12.5</td>
</tr>
<tr>
<td>Platform</td>
<td>Titanium</td>
<td>4520</td>
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<td></td>
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<td>ASME SB265</td>
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</tr>
<tr>
<td>Grade 2</td>
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</tr>
<tr>
<td></td>
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<td>49.8</td>
<td>220.0</td>
<td>11</td>
<td>-5.5</td>
<td></td>
</tr>
<tr>
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<td></td>
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<td>-25</td>
<td></td>
</tr>
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<td>6082 T651</td>
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<td>11</td>
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<td></td>
<td></td>
<td>217</td>
<td>248</td>
<td>8</td>
<td>-15</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5: Density and dimensions of the moveable parts of the mass holder. Three materials have been used. Each of them is a single block piece but here it is decomposed into rectangular section tori whose gravitational contribution can be easily calculated. The position indicated in the last column is relative to the upper surface of the platforms, where the cylinders are positioned.

Vertical velocities of 20 mm/s can be reached. On each driving shaft an encoder (USDIGITAL E6D-1000-250-E1), with a much better resolution (4000 pulses per turn) than that of the motor, is mounted. 1 mm of vertical movement corresponds to 12000 pulses on the encoder. A stepping motor controller QUICKSILVER SILVERNUGGET N2-E1 drives each motor after being programmed through RS-232. In each platform the two controllers are in a master-slave configuration, with the slave tracking the master’s motion.

An optical ruler (HEIDENHAIN LS-603) is fixed on the rigid structure and the pointer is connected to the platform so that the vertical position can be monitored in real-time with a resolution of 1 µm. Reproducibility of the optical ruler readout is excellent, but in order to have 1 µm accuracy on the vertical position it must be aligned to the axis of the structure within 2 mrad.

**Translational degrees of freedom**

The two platforms can be independently and simultaneously moved. The motion is restricted to the vertical axis.
Each platform can be moved by 370 mm, but the two ranges overlap for 187 mm. The vertical position where the two sets of masses can be placed next to each other spans 277 mm (considering also cylinder and platform heights).

**Calibration**

Various tests on the precision and repeatability of the elevator motion were performed. For several heights the platform position was recorded with two independent devices:

- the optical ruler
- an external laser–tracker.

A set of reproducibility and precision movement measurements have been performed on the mass holder and elevator.

The laser–tracker employed is an optical interferometer with a resolution of 1.26 μm (2λ with λ=633 nm). Three cat’s eye holders were glued on each platform, defining points A, B, and C. The laser-tracker can be optically locked on the cat’s eye so that for each platform position it was moved from one holder to another and its position in space was recorded. 6 points on the basement of the holder were used to define a reference plane.

The laser–tracker has an accuracy of ±25 μm, essentially limited by platform vibrations. 100 measurements/s were made.

For holding and lifting up and down a platform two screws on opposite sides are used. The moving systems have been labelled with 1 (master) and 2 (slave) for the lower platform and 3 (master) and 4 (slave) for the upper one. The cat’s eye holders were placed close to axis number 2 (A), axis 1 (B) and 4 (C) on the lower platform and close to axis 2 (A), 3 (B) and 1 (C) on the upper platform.

In this way A and B on the lower platform measure the pitch difference between the two screws and C the deformations from the vertical motion. On the upper platform B acts as a reference position and A and C detect the deformations.

In table 3.6 results originating from a set of 10 movements of 200 mm are reported.

The reproducibility measured with the laser tracker system resulted consistent with the one observed on the optical rulers display within 1 μm.
### 3.3 Source masses and support

<table>
<thead>
<tr>
<th>Platform</th>
<th>Cat’s eye</th>
<th>$\Delta z$ [mm]</th>
<th>$\sigma_z$ [$\mu$m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>upper</td>
<td>$A$</td>
<td>199.9928</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>$B$</td>
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<td>1.3</td>
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<td>$C$</td>
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<tr>
<td>lower</td>
<td>$A$</td>
<td>199.9812</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>$B$</td>
<td>199.9844</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>$C$</td>
<td>199.9914</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 3.6: Results of platform position measurements performed observing their motion with a laser-tracker. A nominal movement of 200 mm was repeated 10 times and the final position was acquired on three points of each platform. $\Delta z$ indicates the measured movement and $\sigma_z$ its standard deviation.

Dynamic measurements were performed as well and a spiral motion with a mean radius of $5 \sim 7 \mu$m and a pitch similar screw’s one was observed.
3.4 Control system

The timing of the experimental sequence must be controlled with a precision better than 100 $\mu$s for most of the experimental actions. A much higher resolution is needed for the interferometer pulse sequence (100 ns).

A software control system was chosen, instead of a hardware one, because of its versatility that is fundamental in applications under development. The control system needs to be programmed in several ways to optimize the experimental sequence and to characterize the apparatus for different parameters values.

The stringent requirement on the timing precision forced us not to work in the user space but in real-time. We opted for a system based on Real Time Application Interface LINUX [73].

The communication between the computer and the instrumentation is done by means of three boards:

- 2 general purpose Input–Output boards Measurement Computing PCI-DAS 1002, each with 24 digital I/O channels, 8 ADC channels (12 bits) used for acquiring experimental data and 2 DAC (12 bits).

- A IEEE-488 control board.

The control program was checked by using two digital lines to trigger the start and stop functions of a HP 53132A counter. Software delays ranging from 100 ns down to 10 $\mu$s were repeatedly (100 times) generated. The measured rms jitter is about 100 ns.

The interferometer sequence of Raman pulses is programmed via GPIB and triggered with a digital channel, but the relative time spacing between the pulses is controlled by the arbitrary wave–function generator driving the AOM, whose time–base is locked to the internal clock of the ANRITSU synthesizer.
Chapter 4

Gradiometer sequence and characterization

In this chapter the experimental sequence adopted to realize a gradiometer measurement is described. Each step is explained and the acquired calibration data are reported and discussed. Particular importance is given to the signal extraction methods. At the end of the chapter some preliminary measurements for characterizing the apparatus and the environmental conditions in which the experiment is realized, are reported.

4.1 Atomic fountain

The first step consists in the implementation of an atomic fountain of cold atoms [74]. This is done by trapping and cooling a sample of $^{87}$Rb atoms with a magneto-optical trap (MOT) [75] and by launching them upwards with a moving optical molasses [76]. Up to $5 \times 10^9$ $^{87}$Rb atoms can be loaded, prepared in the F=2 state and launched upwards with a minimum measured temperature of 4 $\mu$K.

4.1.1 Trapping and cooling - MOT

To explain the trapping and cooling principles [62] let us first consider a 1D system in which a $^{87}$Rb atom is moving with a certain velocity $v$ and two laser beams counterpropagate. If the beams have the same intensity, are equally red detuned and are circularly polarized $\sigma^+$ and $\sigma^-$, because of the Doppler effect the atom has a higher probability of absorbing photons from the beam towards which it is moving. When the atom absorbs a photon
Figure 4.1: (Left) Schematic picture of the MOT in 1-1-1 beams configuration. The six light beams (red arrows) are indicated as well as their circular polarization (white arrows). The two coils producing the quadrupole field are shown together with some corresponding magnetic field lines (black arrows). (Right) Photo of the realized MOT.

from a certain direction it goes into the excited energy level and acquires a momentum quantum, then it spontaneously reemits the photon in a random direction and consequently recoils. The net effect is a viscous force. Let us now add a magnetic field gradient along the axis we are considering. In this way a confining potential having the central point where the magnetic field is zero is created. Extending this principle in the three orthogonal spatial directions the 3D-MOT is obtained as shown in figure 4.1.

Our MOT is realized with three pairs of counterpropagating $\sigma^+/\sigma^-$ polarized laser beams arranged along three orthogonal directions. They are symmetrically tilted from the vertical direction (1-1-1 configuration, see figure 4.1) in order to leave it free for the optical access of the vertical Raman beams. Their frequency can be independently controlled for the three upper beams and for the three lower ones (see section 3.1.3). The MOT beams are collimated and have a waist $w_{\text{MOT}}=11$ mm. Their maximum single beam intensity is 5.5 mW/cm$^2$ (3.3 $I_S$) and for the trapping phase they are all detuned by $-3\Gamma$ from $\nu_{2\rightarrow3}$.

A pair of coaxial coils with opposite currents produces a quadrupole magnetic field. The light beams cross at the center of the trap where the magnetic field is zero. The coils are placed outside the trap chamber and are separated by 16 cm (see figure 4.1). A current $i_{\text{MOT}}$ of 25 A is normally flowing on the series of 50+50 coils with a mean radius of 7 cm producing an axial magnetic field gradient at the center of 8.3 G/cm. The magnetic gradient value along the radial plane crossing the center is one half of the axial one.
In this configuration the slower $^{87}$Rb atoms of the background vapor, that accidentally get into the central region, remain trapped. The maximum velocity $v_c$ an atom can have to be captured depends on the trap parameters in the following way

$$v_c = \sqrt{2w_{\text{MOT}} \Gamma v_r \left( \frac{S}{S + 1} \right)}$$  \hspace{1cm} (4.1)

where $S = I/I_S$ is the saturation parameter and $v_r$ the recoil velocity (see appendix A). For our trap $v_c = 22$ m/s.

The number $N$ of atoms trapped at a certain time $t$ can be determined considering the capture rate $R$ and the loss rate $\Gamma_c$ due to collisions with thermal, not trapped atoms. If we neglect the effect of cold collisions between trapped atoms we have

$$\frac{dN}{dt} = R - \Gamma_c N$$  \hspace{1cm} (4.2)

thus, if $N(0)=0$,

$$N(t) = \frac{R}{\Gamma_c} (1 - e^{-\Gamma_c t}).$$  \hspace{1cm} (4.3)

In figure 4.2 a series of loading curves for different background $^{87}$Rb vapor pressures is shown. We usually keep the rubidium dispenser current
continuously on (operating current: 4.5 A) so that the background vapor level can be more stable.

The atomic density distribution within the trap can be approximately described with a 3D gaussian distribution having a $\sigma$ of 2.5 mm.

4.1.2 Launch

Once a consistent number of atoms, usually about $5 \times 10^8$, is loaded in the MOT the light parameters are changed to further cool the atomic ensemble and to launch it upwards to the desired height. The launching mechanism is based on a moving optical molases [70].

During the loading time the six beams have the same frequency and the atoms are slowed down to a zero mean velocity in the lab reference frame. Considering the beams geometric configuration (1-1-1), if a relative detuning is applied between the three upwards and the three downwards propagating beams the atoms will be still cooled, but the mean velocity will be different from zero because of the Doppler effect.

In practice the upper and the lower beams are detuned by the same frequency amount $\delta$ in opposite directions: $+\delta$ for the upwards propagating beams and $-\delta$ for the downwards propagating ones. Taking into account that the wavevector $k$ of each beam is tilted by an angle $\xi$ ($\cos \xi = 1/\sqrt{3}$) relative to the vertical direction, the atomic cloud can be launched with an initial vertical velocity $v_z$ given by

$$v_z = \frac{\delta}{k \cos \xi}$$

(4.4)

corresponding to 1.35 m/s for each MHz of $\delta$. When the launch sequence begins the quadrupole magnetic field is rapidly (200 $\mu$s) turned off. Simultaneously the beams detuning $\Delta$ is increased to -3.8 $\Gamma$ and the differential detuning $\pm \delta$ is added. After 2.5 ms the detuning is further increased to -6.3 $\Gamma$ and the beams intensity $I$ is decreased, first to a single beam intensity of 1.1 $I_S$ for 1.8 ms, then to $I_S/3$. This step is useful for cooling down the launched atoms since the final temperature scales with $I/\Delta$ [77]. As shown in figure 4.3 the intensity is not abruptly changed, but an RC filter with $\tau = 500 \mu$s was used to smoothen this transition and let the atoms follow the change. In this way lower temperatures of almost a factor 2 could be achieved. A further optimization of the launch parameters should be done in order to reach temperatures slightly above 1 $\mu$K (see for example [78, 79]).
This would allow a gain of more than a factor 10 in the number of atoms contributing to the final signal.

The repumping light (13 mW total power) is kept on during the whole launch sequence and for 3 ms after turning all other lights off in order to pump all the atoms in the $|2\rangle$ state. An experimental efficiency to prepare atoms in the $F=2$ state of more than 99.9% was observed. The total number of launched atoms fluctuates of about 3%.

A good model for describing the atomic distribution $f(\mathbf{r}, \mathbf{v})$ in a cloud expanding after switching off an optical molasses is gaussian both in the space $(x, y, z)$ and velocity domain $(v_x, v_y, v_z)$

$$f(\mathbf{r}, \mathbf{v}) = \frac{N}{8\pi^3} \left( \frac{1}{D^2} \right)^{3/2} \left( \frac{m}{k_B\Theta} \right)^{3/2} e^{-1/2 \frac{mv^2}{k_B\Theta}} e^{-\frac{r^2}{2D^2}}$$

where $N$ is the total number of atoms in the cloud, $D$ is its initial size and $\Theta$ is the temperature of the ensemble, defined as $\Theta = \frac{mv_m^2}{k_B}$. The free time evolution of such a distribution yields an expansion in the three space
dimensions described by the spatial variance \( \sigma \)

\[
\sigma(t)^2 = D^2 + \frac{k_B \Theta}{m} t^2. \tag{4.6}
\]

A much more faithful model that describes the cold atomic sample velocity distribution consists in the square of a Lorentzian distribution \([80]\), but the simpler gaussian model is good enough for our purposes here.

Considering this increasing of the cloud’s radius with time, we performed temperature measurements by simply launching the cloud upwards and determining its size twice, when going upwards and when falling down at a certain height \([81, 82]\). A slightly red detuned horizontal thin sheet of light was used to shine the atoms and the fluorescence emitted was detected as a function of time.

A minimum temperature of 4 \( \mu \)K was observed. The launch sequence parameters and the bias magnetic fields were experimentally optimized in order to reach the minimum temperature. By turning on the MOT again while the cloud falls, it is possible to recapture part of the atoms launched. When the atoms fall back in the trap, their velocity is obviously the same as the launch initial velocity (1 m high means \( v_z = 4.4 \) m/s), therefore much
smaller than \( v_c \). The recapture efficiency is reported in figure 4.4 as a function of the launch height \( h \), that is related with the time of flight in the fountain \( t_f \) by \( t_f = \sqrt{8h/g} \). The loss is mainly due to the expansion of the cloud. In fact at 4 \( \mu \)K (equivalent to 12 times the recoil temperature \( T_r \) reported in appendix A) the thermal velocity \( v_r \) is 20 mm/s (3.5 \( v_r \)), so after a time of 1 s, the cloud radius increases by 20 mm. This is larger than the capture radius of the MOT, determined by the magnetic field gradient and by the beams diameter. About 126 mm above the MOT there is the upper window of the vacuum system and atom that reach such a height thermalize, therefore we observe the rapid loss of atoms \cite{83}.

4.2 Juggling

For the gradiometer we need two cold atomic samples falling with the same velocity, and vertically separated by 30 ~ 40 cm. The two clouds should contain a large number of atoms \((10^8 ~ 10^9)\) and must be launched in rapid sequence (within 100 ms). Both requirements have to be fulfilled because on the one hand \(10^5 ~ 10^6\) atoms per cloud must remain after internal state and velocity selection (see section 4.6) and on the other hand the two clouds are launched 60 and 95 cm above the MOT because of the apparatus geometry. It is not possible to satisfy both requirements by simply loading atoms from the background vapor since the loading rate is not high enough.

Two possible solutions were considered:

**Juggling** \cite{84} - A cloud of atoms is loaded and launched upwards. During its time of flight another cloud is loaded. Just before the falling atoms reenter the trap region the other cloud is launched. The first cloud is then recaptured with high efficiency (see figure 4.5).

**2D–MOT** \cite{85, 86} - A continuous flux of cold atoms, created with a two-dimensional MOT, can be directed to the 3D-MOT to quickly reach the desired number of atoms.

While building up and characterizing the 2D-MOT system in a separate apparatus that will be connected to the atomic fountain when ready, the first solution was adopted.

In figure 4.5 the fluorescence signal detected in the trap chamber during a juggling sequence is shown. An atomic cloud of \(5\times10^9\) atoms is loaded in 3 s and launched up to 60 cm so its time of flight will be 700 ms. About 50
ms later the MOT is turned on again and the loading of a new cloud begins. This can only last 570 ms because the first cloud is falling back. At that time the new cloud is launched up to 95 cm. 80 ms after this launch 25% of the atoms of the first cloud is recaptured, cooled and relaunched up to 60 cm, within only 25 ms. For about 20 ms after a launch MOT magnetic fields can not be turned on because atoms close to the trap would be deflected by the magnetic field gradient.

In this way two samples of $5 \times 10^8$ atoms are launched 60 and 95 cm above the trap within 100 ms. With this method and in this geometric configuration the number of atoms in both clouds can not be further increased significantly if not by increasing the background rubidium pressure, that would though introduce collisional problems. The second solution (2D-MOT) will be then adopted as soon as the system will be ready.

4.3 Internal state and velocity selection

The cloud launched in the fountain is a heterogeneous sample of atoms in terms of internal state and velocity distribution (see figure 4.6).
• 99.9% of the sample is in the F=2 state, but a residual 0.1% is still in the F=1 state,

• they are distributed among all the magnetic sublevels,

• a HWHM $\Delta v = v_T = 20$ mm/s along each direction describes the velocity distribution and determines cloud’s expansion in time.

The interferometer works independently on each single atom. One would like all the atoms to give identical signals that add constructively avoiding contrast loss.

Atoms in different magnetic sublevels interact in different ways with a magnetic field and different velocities imply spatial spread, thus different Doppler effect and different gravitational interaction with surrounding masses.

For these reasons an internal state $|F,m_f\rangle = |1,0\rangle$ and a vertical velocity class $\Delta v$ selection is implemented after the launch. The following selection sequence, illustrated in figure 4.6, was chosen and it is realized when the atoms are in the magnetically shielded vacuum tube where a bias magnetic field of 250 mG is applied using the long coil (see section 3.2.2).

• First, the fraction of atoms remained in the F=1 state after the launch is further reduced using a 10 ms light pulse resonant with the transition F=1→F’=0. The beam is divergent, elliptically polarized and is slightly tilted from the vertical direction in order not to be retrore-
flected. A blow-away efficiency of about 50% was measured, limited by the pumping into dark states in F=1.

- A velocity selective Raman π pulse (I_{tot}=100 mW, τ=100 μs) is then vertically applied [87]. The laser beams frequency difference is resonant with the transition |F=2,m_f=0⟩ → |F=1,m_f=0⟩), so a selected velocity class of atoms (that also has a confined vertical distribution) is transferred into the |F=1,m_f=0⟩ state. As discussed in section 2.2, the Doppler shift of the transition frequency is δ_R^{Doppler} = p·k_L/m = ν·k_L. Therefore a velocity spread Δv_z of the selected atoms is related to the width of the Raman transition Δδ_R^{Doppler} by

\[
Δv_z = \frac{Δδ_R^{Doppler}}{k_{eff}}. \tag{4.7}
\]

The Raman linewidth itself is very small, because ω_{eff} is accurately stabilized and because it involves two states with a very long lifetime. What limits the linewidth in our case is the interaction time with the Raman light. For τ =100 μs the Fourier transform of the pulse has a FWHM approximately of 10 kHz, so Δv_z ≈ 3.9 mm/s. It is useful to compare the HWHM velocity distribution after the selection and the recoil velocity

\[
Δv_z = 1.95 \text{ mm} \text{ s}^{-1} \quad \text{v}_r = 5.9 \text{ mm} \text{ s}^{-1}.
\]

These atoms have a velocity distribution HWHM of v_r/3 (corresponding to T_T/9=40 mK) along the vertical direction, whereas remain with a velocity distribution HWHM Δv of 3.5 v_r along the horizontal directions.

- Remaining atoms in the F=2 state are blown away by means of a 5 ms pulse (slightly divergent, circularly polarized) tuned to ν_{z-3}.

The remaining sample contains only 10^5 atoms in the |F=1,m_f=0⟩ state and has a disk-shaped spatial distribution, wide in the horizontal directions (no selection) and very thin in the vertical direction (about 1/10) thanks to the velocity selection.

The number of atoms has been reduced by a factor 500. A significant efficiency enhancement could be obtained implementing methods that allow not to just select the desired sample of atoms, but preparing many atoms
in the wanted state. For example a more effective cooling mechanism could be implemented to increase the fraction of atoms in the velocity group addressed by the Raman pulse, and RF pumping could be used to populate the F=2 m_e=0 state.

4.4 Single Raman pulse - Rabi oscillations

After the selection the atomic ensemble is freely falling in the vacuum tube. If now a Raman pulse of length τ is applied an electric dipole moment is induced on each independent atom, as explained in section 2.2.

The amplitude probabilities in the single atom wavefunction after the atom–Raman light pulse interaction for a time τ do not vary (only the relative phase between the two states does). It is possible to experimentally measure the atomic internal state. The single measurement is just a projection of the quantum state so we will find the atom either in state F=1 or in F=2 with probabilities \( P_1 \) and \( P_2 \). Since we can use a large amount of atoms, that in principle should be characterized by the same amplitude probabilities, we make a statistical measurement and therefore it is correct to associate the measured fraction of atoms in F=1, \( n_1 = N_1/N_{\text{tot}} \), to the probability \( P_1 \) of the single atom.

This procedure is usually limited by the quantum projection noise (QPN) \cite{88, 89}, that scales as \( \sqrt{N_{\text{tot}}} \). The intrinsic limit to signal to noise ratio \( S/N \)
in the measurement, for this reason, is proportional to $|\frac{S}{N}|_{QPN} = \sqrt{N_{tot}}$. This suggests to use a larger atom number. Increasing it too much, though, implies either contrast loss due to different atomic signals, if they are spatially spread, or systematic shifts related to atom–atom interactions, if they are a dense ensemble. Usually $10^6$ atoms, providing a theoretical $|\frac{S}{N}|_{QPN} = \sqrt{N_{tot}}=1000$ are considered a good result.

In figure 4.7 Rabi oscillations are shown. For each Raman intensity set the pulse length was varied from 10 µs to 500 µs in steps of 10 µs. The oscillations are strongly damped because longer interaction times select a narrower velocity class. The mean level is 0.25 because the single pulse itself is not 100% efficient on the atomic ensemble. In the standard experimental conditions π pulses of τ =100 µs are used.

### 4.5 Three pulse sequence - Atom Interferometer

The interferometer sequence is composed of three pulses ($\frac{\pi}{2} - \pi - \frac{\pi}{2}$) spaced by a time $T$ [28]. As discussed in section 3.1.5 the Raman beams frequency difference is linearly swept for compensating the gravitational Doppler shift and only one pair of beams ($\omega_{R1 \uparrow}, \omega_{R2 \downarrow}$) is resonant with the atomic frequency in its reference frame. When the atom is at rest, i.e. at the turning point of the ballistic flight, there is no difference between the two pairs of Raman beams in terms of frequency. The difference is then just the sign of $k_{\text{eff}}$. Half of the atoms would exchange the wrong momentum and their contribution to the signal would be lost. On the other hand, if the central π pulse is sent when the atoms invert their motion then the interferometer is symmetric and spatial effects like magnetic field gradients would be completely compensated during the two interferometer parts.

Experimentally the π pulse is sent close enough to the inversion point to symmetrize as much as possible the two interferometer arms, but far enough so that the atom has a velocity component sufficient to neglect the second pair of Raman beams. In our case the π pulse is 5 ms after the turning point when the velocity is 49 mm/s. At that time the atoms are only 123 µm below the turning point and the uncompensated path is 15 mm (for $T=150$ ms).

Taking into account also the beams polarization and the Zeeman effect induced by the vertical bias field, only Raman transitions with $\Delta m_{F} = 0$ are possible because the combinations of the Clebsch–Gordan coefficients let other possibilities vanish. Considering that the quantization axis defined by
the vertical magnetic bias field is parallel to $k_{\text{eff}}$, it is possible to express each linearly polarized Raman field as a linear combination of right and left circularly polarized light

$$E \uparrow = \sigma^+ + \sigma^- \quad E \downarrow = \sigma^+ - \sigma^-$$

(4.8)

where the minus sign in the downwards propagating light comes from the double pass through the quarter-wave plate above the vacuum system. In this way $\pi$ transition are suppressed.

Finally, of the 18 combinations reported in table 4.1, the transitions including $\pi$ polarized photons, those with wrong energy ($\Delta E_x \neq 0$) and obviously those with vanishing probability ($P_{\text{rel}} = 0$) are eliminated. Only

$$|F = 1; m_\nu = 0 \rangle \rightarrow |F = 1, 2; m_\nu = \pm 1 \rangle \rightarrow |F = 2; m_\nu = 0 \rangle$$

are allowed and contribute to the desired Raman transition.

In section 2.3 the interferometer phase term induced by local gravity is $\Delta \phi_g = k_{\text{eff}} g T^2$. When repeating the measurement one obtains the same result if no other effect contributes. So a controlled phase shift $\phi_{\text{ext}}$ is experimentally introduced between the $\pi$ pulse and the last $\pi/2$ pulse in order to scan the interferometric fringes. Figure 4.8 shows the interferometer fringe obtained with $T = 5$ ms, corresponding to the fraction of atoms detected in the $F=1$ state normalized to the total number of atoms detected in each launch. In this case the phase shift of the 6.8 GHz synthesizer (see section 3.1.5) is varied in steps of $2^\circ$ from launch to launch.
The sinusoidal distribution of points was fitted with three parameters, the amplitude $A$, the phase offset $\theta$ and the vertical offset $B$

$$f(\phi_{\text{ext}}) = A \sin (\phi_{\text{ext}} + \theta) + B$$

resulting in a phase resolution of 6 mrad. Considering that each shot took 5.83 s, the phase sensitivity for this measurement was 209 mrad/Hz. The fringe contrast defined as $A/B$ in this case is 26%.

### 4.6 Detection scheme

The atomic wavefunction after the interferometer is a superposition of the two states $F=1$ and $F=2$ with probability amplitudes containing information about the acceleration impressed to the atoms and any other dephasing effect. The detection of the relative population of the two states can be therefore connected to the acceleration.

The detection is realized by first measuring the number of atoms in the $F=2$ state, then these atoms are blown away and afterwards the remaining ones in the $F=1$ state are measured as well. Two rectangular shaped (5 mm $\times$ 12 mm), horizontally aligned, retro-reflected beams are used, vertically displaced by 20 mm one above the other, as shown in figure 4.9. Falling atomic clouds pass first through the upper one and the fluorescence emitted, proportional to the number of atoms in the state $F=2$, is collected by a $f=50$ mm, 2" diameter lens on the lower photodiode. The lower part of the upper detection beam is not retro-reflected and the detected atoms are pushed out of the lens focus. The lower beam is used for the detection of the remaining atoms in $F=1$. Their fluorescence is detected on the upper photodiode. An optical filter (ANDOVER CORPORATION 780FS10-50) is placed in front of the photodiodes to reduce the percentage of detected non resonant light.
A horizontal magnetic field is provided with external coils. Circularly polarized light, with respect to this magnetic field axis, is used in order to cycle on the closed transition $|F = 2; m_F = 2\rangle \rightarrow |F = 3; m_F = 3\rangle$ avoiding substantial losses through dark states. $I_S$ for such a light on rubidium is 1.67 mW/cm$^2$. The detection light, slightly red detuned from the transition ($\nu_{2\rightarrow3} = 800$ kHz), is split in the two equally intense (0.6 mW/cm$^2$) beams. On the lower one a low intensity (200 $\mu$W/cm$^2$) repumping beam is added in order to pump the atoms from the F=1 into the F'=2; they spontaneously decay in F=1 or F=2, but only the latter is a dark state for the repumping light and the effective pumping from F=1 to F=2 is so realized. At this point the detection is exactly the same as for the upper beam.

Two 10 × 10 mm photodiodes (Hamamatsu S1722–05) are used. A 1 GΩ transimpedance amplifier with an OPA627 converts the photodiode current into a voltage. The bandwidth is of the order of 1 kHz and the current noise density referred to the input is 15 fA/$\sqrt{\text{Hz}}$ limited by the shot noise dark current in the photodiode. Smaller photodiodes are expected to reduce the current noise density at about 5 fA/$\sqrt{\text{Hz}}$ limited by the Johnson noise in the 1 GΩ resistor.

The signal is further amplified by a factor 70 and the offset due to stray light and dark current removed to match the dynamic range of an analog-to-digital 12 bits converter.

An example of the atomic fluorescence signal acquired with this detection setup is reported in figure 4.10. The two peaks are separately detected by
the two photodiodes as a function of time while the atomic cloud is falling through the detection beams. The time delay is due to the time of flight between the two beams height. The S/N relative to the peak detection is 70/1.

An atom, lying in a standing wave with a total intensity $I$ and detuning $\Delta$ scatters photons with a rate of

$$R_{sc} = \frac{\Gamma}{2} \left( \frac{I/I_S}{1 + 4(\Delta/\Gamma)^2 + I/I_S} \right)$$

and the total power emitted by the atom is then $P = h\nu_{2\rightarrow3}R_{sc}$. A photodiode with quantum efficiency $\eta_{PD}$ that receives the fluorescence light emitted by $N$ atoms within a solid angle $\Omega$ detects

$$P_{PD} = \eta_{PD} \eta_f \frac{\Omega}{4\pi} N h\nu_{2\rightarrow3} R_{sc}.$$ 

where $\eta_f$ is the transmittivity of the optical filter (55%). The solid angle in our detection scheme is $\Omega/4\pi = 7 \times 10^{-3}$ and the photodiode responsivity $\eta_{PD} = 4.5 \times 10^8$ V/W. The sensitivity to the atomic fluorescence is about $4 \times 10^5$ atoms/V.
4.7 Gradiometer sequence summary

The whole experimental sequence is summarized in figure 4.11. The juggling technique allows to launch two laser cooled samples of $5 \times 10^8$ atoms in rapid sequence. They are vertically displaced by 35 cm and travel with the same average velocity (as indicated by the two shifted parabolas). Once both samples are in the magnetically shielded tube a Raman pulse is used to select a narrow velocity class in the broad distribution. Then the three-pulse interferometer sequence is applied, slightly delayed from a symmetric configuration around the atomic turning point. All Raman pulses act on the two samples simultaneously and in the same way (densities are too low to observe shadowing effects). On their way back down the atoms pass through two horizontal beams and the fluorescence corresponding to both states are detected, first for the lower cloud and then for the upper one.
4.8 Signal extraction

The detected signal must be eventually reported in terms of relative population in the two atomic hyperfine states for both ensembles. From each photodiode a signal showing two peaks (see figure 4.9), corresponding to the passage of the two clouds, is acquired. Such signals are fitted with gaussians and a background linear slope. The free parameters in the fit are three for each gaussian (amplitude, position and width) and two for the linear slope.

The area of each gaussian signal ($S_F^n$) resulting from the fits is proportional to the number of atoms in the corresponding state ($F = 1, 2$) and cloud ($z \text{=up,dw}$), as can be deduced by equation (4.11). For each cloud the fraction of atoms in each state is thus obtained by

$$
n_1^{up} = \frac{S_1^{up}}{S_1^{up} + S_2^{up}} \\
n_2^{up} = \frac{S_2^{up}}{S_1^{up} + S_2^{up}}
$$

$$
n_1^{dw} = \frac{S_1^{dw}}{S_1^{dw} + S_2^{dw}} \\
n_2^{dw} = \frac{S_2^{dw}}{S_1^{dw} + S_2^{dw}}
$$

(4.12)

Fringes can be obtained plotting one of these normalized numbers of atoms if an external phase scan is applied from launch to launch.

4.8.1 Sine fitting

In the gradiometer configuration one is interested in measuring the differential phase accumulated by the two atomic samples during the simultaneous interferometers. If the phase noise is not too high the phase shift can be extracted by directly fitting the fringes. Some examples are reported in figure 4.12 where lower cloud's (left) and upper cloud’s (right) fringes are reported for $T = 5, 50$ and $150 \text{ ms}$. Recalling equation (2.46) accelerations are detected proportionally to the square of the interferometer time $T$. In a gravimeter, the acceleration is measured with respect to the reference frame defined by the upper mirror. Mechanical vibrations of the mirror are then eventually converted to phase noise in the fringes. The spectral density of this mechanical noise is responsible for the rapid deterioration of visibility in figure 4.12.

Here the sensitivity to the phase $\sigma_\phi$ and the contrast resulting from the sinusoidal fit for each independent fringe are reported.
Figure 4.12: Fringes relative to the lower (left) and upper (right) interferometers are reported for $T = 5, 50$ and 150 ms (from top to bottom). While contrast remain approximately the same in the different configurations, fringe visibility is rapidly lost as $T$ increases because of the increasing sensitivity to mirror vibrations. For $T = 5$ and 50 ms the data are fitted with a sinusoidal function.

<table>
<thead>
<tr>
<th>$\sigma_\phi$</th>
<th>up</th>
<th>$A/B$</th>
<th>dw</th>
<th>up</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.19 rad/√Hz</td>
<td>0.26 rad/√Hz</td>
<td>28%</td>
<td>28%</td>
<td></td>
</tr>
<tr>
<td>1.1 rad/√Hz</td>
<td>1.1 rad/√Hz</td>
<td>23%</td>
<td>24%</td>
<td></td>
</tr>
<tr>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>
4.8.2 Ellipse fitting

The powerful method of using two freely falling samples and detect their acceleration by means of the same Raman lasers lies in the fact that mirror vibrations are a common mode effect and vanish in the differential measurement. What is detected is then the acceleration of one sample relative to the other one that are vibration free because both clouds are freely falling in vacuum. If the upper fringe is plotted versus the lower one, Lissajous curves are obtained and the phase difference can be extracted by fitting the point distribution with an ellipse [90].

The two fringes can be described with independent amplitudes ($A$ and $C$) and vertical offsets ($B$ and $D$) and with a relative phase difference $\phi$ that is the parameter of our interest

$$\begin{align*}
x &= A \sin (\theta) + B \\
y &= C \sin (\theta + \phi) + D
\end{align*}$$

A simple translation removes the offset terms $B$ and $D$

$$\begin{align*}
x' &= A \sin (\theta) \\
y' &= C \sin (\theta + \phi)
\end{align*}$$

leading to the ellipse equation

$$\frac{x'^2}{A^2} + \frac{y'^2}{C^2} - x' y' \left( \frac{2 \cos \phi}{AC} \right) - \sin^2 \phi = 0.$$
Let us now apply a rotation in order to eliminate the \( x'y' \)-term and obtain
the ellipse equation in normal form

\[
\begin{align*}
  x'' &= x' \cos \alpha + y' \sin \alpha \\
  y'' &= -x' \sin \alpha + y' \cos \alpha
\end{align*}
\]  

(4.16)

with

\[
\tan 2\alpha = \frac{2AC \cos \phi}{C^2 - A^2}.
\]  

(4.17)

Finally we have

\[
\frac{a'^2}{a^2} + \frac{b'^2}{b^2} = 1
\]  

(4.18)

with

\[
\begin{align*}
  a^2 &= \frac{2A^2C^2 \sin^2 \phi}{A^2 + C^2 + \sqrt{A^4 + C^4 + 2A^2C^2 \cos 2\phi}} \\
  b^2 &= \frac{2A^2C^2 \sin^2 \phi}{A^2 + C^2 - \sqrt{A^4 + C^4 + 2A^2C^2 \cos 2\phi}}
\end{align*}
\]  

(4.19)

Besides the offset parameters \( B \) and \( D \) we obtained a canonic expression
with \( a(A, C, \phi) \) and \( b(A, C, \phi) \) with a transformation parameter \( \alpha(A, C, \phi) \).

For finding the "best fitting" ellipse in canonical form to a set of \( n \) experimental data points \((x_i, y_i)\) we minimize the \( \chi^2 \)

\[
\chi^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x}_i)^2}{\sigma_{x_i}^2} + \frac{(y_i - \overline{y}_i)^2}{\sigma_{y_i}^2}.
\]  

(4.20)

where \((\overline{x}_i, \overline{y}_i)\) is a point on the ellipse. Assuming \( \sigma_{x_i} = \sigma_{y_i} = \sigma \) we can reduce the problem to a least squares one and solve it applying the method of the Lagrange multipliers minimizing

\[
S = \sum_{i=1}^{n} (x_i - \overline{x}_i)^2 + (x_i - \overline{x}_i)^2 - \lambda_i f(\overline{x}_i, \overline{y}_i)
\]  

(4.21)

with respect to \( a, b \) and all the \( \overline{x}_i, \overline{y}_i, \lambda_i. f(\overline{x}_i, \overline{y}_i) = 0 \) is the constraint that
the points \((\overline{x}_i, \overline{y}_i)\) must lie on the ellipse, i.e. equation (4.18).

The adopted algorithm can be summarized in this way:

- A set of parameters \([A, B, C, D, \phi]_k\) is used and the experimental points
  \((x_i, y_i)\) are all roto–translated (see transformations (4.14) and (4.16))
  into a new distribution of points with a canonic ellipse shape.

- \( S_k \) is determined.
- A numerical tool (MINUIT) is used to find the minimum value of the multi-parameter function $S$. A new set of parameters $[A, B, C, D, \phi]_{k+1}$ is produced and the procedure is repeated until $S$ is minimized.

- The desired differential phase between the two interferometer fringes is thus directly the parameter $\phi$.

### Sensitivity to noise

In order to understand what kind of noise is dominant in the experimental set of data three different simulations are reported here. In similar conditions to the experimental ones ($A=0.13, \ C=0.11, \ B=D=0.5, \ \phi=1.0 \ \text{rad}$) ellipses were simulated for three values of offset noise $\sigma_o$, amplitude noise $\sigma_a$ and phase noise $\sigma_\phi$ added on each point.

**Offset noise**

\[
x = A \sin \theta + B + \sigma_o \\
y = C \sin (\theta + \phi) + D + \sigma_o
\]  

(4.22)

**Amplitude noise**

\[
x = (A + \sigma_a) \sin \theta + B \\
y = (C + \sigma_a) \sin (\theta + \phi) + D
\]  

(4.23)

**Phase noise**

\[
x = A \sin \theta + B \\
y = C \sin (\theta + \phi + \sigma_\phi) + D
\]  

(4.24)

The three different kinds of noise produce qualitatively different ellipses. Offset noise, that can be induced for example by a fluctuation of the repumper light frequency, moves the ellipse around in the $xy$ plane. Amplitude noise can be induced by the detection or by variations of the effective two-photon Rabi frequency that implies not perfect $\pi$ and $\pi/2$ pulses. It is amplified correspondingly to the maxima and minima on the $x$ and $y$ axis and minimized between those four points. The opposite happens in case of phase noise as clearly visible in the simulations at the bottom of figure 4.14. Residual phase noise could be introduced by local effect fluctuations like spurious electromagnetic fields.

A comparison between simulated elliptic point distributions with experimentally acquired ones shows that the dominant noise in the present experimental situation is related with amplitude fluctuations.

100 ellipses with different amplitude noise, ranging from 1% to 10%, were generated for different $\phi$ values. Each of them was fitted with the
ellipse fitting method and the difference between the set differential phase in the simulation and the one obtained from the fit as a function of $\phi$ is shown in figure 4.15. Because of the noise, a bias is present on the determination of the right phase term. The present experimental amplitude noise is a few percent so the error in the determination of the phase with the ellipse fitting routine is between 10 and 20 mrad for very small phase differences, but is decreased to about 1 mrad if the relative phase is around $\pi/2$, as we do in the experiment.
4.9 Measurements

Before coming to the measurement of the gravitational constant some characterization measurements of the apparatus and its performances [91] are briefly reported in this section. These measurements are useful for accurately estimate the systematic errors in the $G$ determination itself.

4.9.1 Velocity distribution

We have measured the vertical velocity distribution of the atoms after the Raman velocity selection pulse. The procedure used consisted in selecting the atoms as described in section 4.6 and then, after 100 ms by applying a second Raman pulse to transfer a certain amount of atoms back into the empty $F=2$ state. The frequency difference between the Raman lasers for the second pulse was scanned across the resonance (in the atomic frame) in order to select different velocity classes of the atomic distribution.

From a gaussian fit, also reported in figure 4.16, on the acquired velocity distribution a HWHM of 0.5 $v_r$ was measured.

4.9.2 Magnetic field mapping

The magnetic field present along the central axis of the interferometer tube in the common experimental conditions was measured directly with atoms. With the procedure described in section 4.9.1 a frequency scan on
4.9 Measurements

Figure 4.16: Vertical velocity distribution of the atomic sample after Raman velocity selection. The selection pulse was applied always with same frequencies and at the same time. Then a second pulse with Raman frequency difference varying from launch to launch scans different atomic velocity classes.

Figure 4.17: Magnetic field mapping along the central axis of the interferometer tube. The reported data indicate the differential effect observed on atoms in the $m_p = 1$ and in $m_p = 0$ state. The $z$ origin is approximately in the MOT region. The bias field long coil was continuously kept on and a magnetic field was pulsed powering two of the ten short coils. The theoretical curve is reported as well.

the second pulse was repeated for several heights in the fountain. The fit on each scan could provide a measurement of the resonant frequency as a function of the vertical position. The whole tube length was scanned using atoms in the $m_p = 0$ state. The same scan was then repeated using atoms in the $m_p = 1$ state. By subtracting the results a map of the magnetic field was obtained and it is reported in figure 4.17. During the measurement the bias field long coil was continuously kept on and a localized magnetic field
Figure 4.18: (Left) Doppler frequency shift for atoms in the $m_F=0$ state during the free fall in the fountain. For any given height the reference oscillator frequency resonant with the atoms and resulting from a gaussian fit on the frequency scan is reported. The linear fit on the data set gives a slope proportional to the local gravity. The obtained value for $g$ is $9.8056(1)$ m/s$^2$. (Right) Residuals of the linear fit.

was pulsed using two short coils out of the series of ten (second and third coils starting from the bottom, see section 3.2.2), exactly as it is usually done during the experimental sequence.

The magnetic field was measured along the central axis of the shielded tube with a resolution of 100 μG. The fluctuations that can be seen on the top of the tube in figure 4.17 are real spatial fluctuations of the field. Repeating the measurement, in fact, the results did not change by more than 100 μG for any given vertical position.

### 4.9.3 Measurement of local $g$ by Raman velocimetry

The frequency scans around the atomic resonance for any given height in the fountain were also interpreted as a two-photon Raman velocimetry determination of local gravity with atoms in the $m_F = 0$ state. In figure 4.18 the resonant frequency resulting from the gaussian fit for each vertical position is reported. Residuals from the linear fit are not structured. A value of local gravity $g = 9.8056(1)$ m/s$^2$ was obtained with this method. The results obtained with the two clouds are consistent, their difference lies well below the reported uncertainty.

The vertical alignment of the Raman beams was monitored with a tiltmeter on their retroreflecting mirror. The beams vector was vertical within an angle $\theta = 20 \mu$rad corresponding to a systematic reduction of real local $g$ by less than $2 \times 10^{-10}$ due to misalignments.
4.9 Measurements

![Graph showing phase shift vs. coil current]

Figure 4.19: (Left) Phase shift resulting from the elliptic fit for different current values flowing in the short coils number 3 and 4. The parabolic curve indicates the second order Zeeman effect on the atoms in $m_f=0$. The value resulting from the fit for $i = 0$ is the residual phase difference between the two displaced interferometers. (Right) Three examples of elliptic point distributions for current values $i=3,5,7$ mA in the short coils number 3 and 4.

4.9.4 Magnetic pulse detection

For the reasons explained in section 4.15 small differential phases are not accurately fitted with the ellipse fitting method so it is convenient to add an external, well-controlled relative shift in order to have a differential phase of about $\pi/2$. An easy way to do that is to apply a localized uniform magnetic field pulse during the second part of the interferometer, only on one of the two atomic samples. As described in section 2.3.2, during the pulse the atomic energy levels are shifted by the second order Zeeman effect, in different ways for the two clouds and an additional phase shift is then acquired. As a function of the current in the coils, the phase shift resulting from the elliptic fit is reported in figure 4.19 with an interferometer time of 100 ms. Some ellipses are shown as well. The long coil producing the bias magnetic field was on, as always, therefore the differential shift was proportional to the difference between the squares of the fields in the two interferometric regions ($B_{dw}^2 - B_{up}^2$).

The extrapolation of the zero crossing with a parabolic fit, also reported in figure 4.19, leads to a non zero shift that is to first order induced by the Earth gravity gradient. The phase value resulting from the fit is 128(5) mrad and the corresponding gradient differs from the expected gravity gradient by 24%. Possible systematic shifts could be induced in this situation by a non vertical launch of the atomic cloud (see section 2.3.1).
4.9.5 Measurement of the Earth’s gravity gradient

A preliminary measurement of the Earth gravity gradient was performed by realizing a long series of gradiometric measurements. A total 99 cycles from 0° to 355° in steps of 5° were done during a whole night (7118 measurements - about 10 hours). The sequence was always the same. No magnetic field pulse was used to further shift the relative phase and the cylinders were removed from the platforms in order to minimize the induced systematic errors. The effect of the platforms was taken into account in the simulation while the gradient induced by the rest of the apparatus was estimated to be negligible. The two clouds were separated by 374.3 mm and $T$ was set to 150 ms. Each cycle of 72 differential measurements created an ellipse independently fitted. The resulting phases are reported in figure 4.20.

The error on each single point is estimated by introducing the constraint that the reduced $\chi^2$ is equal to 1. From that we determine $\sigma_\Delta\phi$.

The expected Earth gravity gradient $(3.08(1) \times 10^{-6} \text{ s}^{-2})$ shift is (see section 2.3) $\Delta\phi_{\text{grad}} = 417.8$ mrad. From place to place though it varies because of the effect of near masses. The biggest contribution is given by the platforms, their total effect increases the phase difference by 19.2 mrad. With a rough estimate, then, the optical table (NEWPORT RS 2000), with its weight of 340 kg adds a differential gravitational acceleration between the two clouds of $GM_{\text{table}}(\frac{1}{h_{\text{cloud}}} - \frac{1}{h_{\text{platform}}}) \sim 10^{-8} \text{ m/s}^2$, meaning an effect of the order of 4 mrad. Furthermore the misalignment of the launch direction, as
explained in section 2.3, has to be considered since it can add a Coriolis shift.

The results obtained are reported in figure 4.20. The weighted average value was $\Delta \phi = 470.8 \text{ mrad}$ with a standard deviation of $\sigma_\phi = 2.3 \text{ mrad}$. Therefore the statistic error was 0.5% and, regarding accuracy, the expected value differs from the obtained one by 8%, without considering the vertical launch misalignment. A tilt of 1.1 mrad could induce a shift of 8% on the measured differential value.
| $|F = 1\rangle$ | Abs | $|F = 2\rangle$ | $\Delta E_z$ | $P_{\text{rel}}$ |
|---|---|---|---|---|
| $|m_F\rangle$ | $|m_F\rangle$ | $\Delta m_F$ | |
| $\sigma^-$ | +2 | 1 | -3 | $10\sqrt{6}$ |
| $\sigma^+$ | +1 | 0 | -2 | $10\sqrt{3}$ |
| +1 | $\pi^-$ | +1 | -3 | $-1 \times 10\sqrt{6}$ |
| $\pi^+$ | +1 | 0 | -2 | 0 |
| $\sigma^+$ | 0 | -1 | -1 | 10 |
| 0 | $\sigma^-$ | +1 | 0 | -2 | $-1 \times 10\sqrt{3}$ |
| $\sigma^+$ | +1 | 1 | -1 | $-10\sqrt{3}$ |
| $\pi^+$ | 0 | 0 | 0 | $-20$ |
| $\sigma^+$ | 0 | 0 | 0 | $-1 \times 20$ |
| $\pi^-$ | +1 | 1 | -1 | $10\sqrt{3}$ |
| $\sigma^+$ | -2 | -2 | +2 | 0 |
| -1 | $\sigma^-$ | +1 | 2 | 0 | $-1 \times 0$ |
| $\sigma^+$ | 0 | 1 | +1 | $-10$ |
| $\pi^+$ | 0 | 0 | +2 | $-10\sqrt{3}$ |
| $\sigma^-$ | 0 | +1 | +1 | $-1 \times -10\sqrt{3}$ |
| $\pi^+$ | -1 | 0 | +2 | 0 |
| $\sigma^-$ | -2 | -1 | +3 | $10\sqrt{6}$ |
| $\sigma^+$ | -1 | 0 | +2 | $-1 \times -10\sqrt{3}$ |
| $\pi^+$ | -2 | -1 | +3 | $-10\sqrt{6}$ |

Table 4.1: Raman transitions from $F=1$ to $F=2$ of the $^{87}$Rb ground state $5^2 S_{1/2}$. Absorption (Abs) and Emission (Em) indicate the kind of transition stimulated by laser 1 and 2, respectively. $\Delta E_z$ is the Zeeman shift of the transition frequency in multiples of the Zeeman splitting $\Delta E = N \cdot \mu_B g_F B_z$. Since our beams are parallel to the B-field, the $\pi$-transitions are suppressed. The last column specifies the relative transition probabilities $P_{\text{rel}}$ multiplied by 120 (see figure A.1), which are determined by the Giebisch–Gordan coefficients. The probabilities are calculated for Raman beams in a lin-1lin configuration, for which the minus-sign of the second $\sigma^-$ transition is added. To obtain the total probability for a transition $|F = 1,m_F\rangle \rightarrow |F = 2,m_F\rangle$, one has to sum the probabilities of all possible transitions between these two states.
Chapter 5

Measurement of $G$

This chapter is dedicated to the measurement of the Newtonian gravitational constant. After a preliminary description of the experimental configuration the methods used for the computer simulation of the experiment are described.

The preliminary results obtained first using lead cylinders are presented. Then the $G$ measurement performed with the final tungsten source masses is reported.

5.1 Experimental configuration

All the parts of the apparatus and the experimental methods described so far have been developed aiming at the realization of a $G$ measurement.

The cylinders have been placed on the two platforms as shown in figure 3.8. In the first configuration, labelled C1, the two sets of masses are vertically positioned next to each other, while in the configuration C2 they are far apart. The vertical positions of the cylinders’ center of mass are labelled with $H_{\text{dw}}^{C1}$, $H_{\text{up}}^{C1}$, $H_{\text{dw}}^{C2}$ and $H_{\text{up}}^{C2}$, whereas the two clouds inversion points are indicated with $h_{\text{dw}}$ and $h_{\text{up}}$. The zero of the vertical axis is set at the upper surface of the base holding the structure of the mass support (see figure 3.12).

The $G$ measurements have been performed with an interferometer time $T$ set to 150 ms and a delay $\Delta t = 5$ ms between the time of the atomic trajectory inversion point and the time of the central $\pi$ pulse, as also shown in figure 5.1. Consequently the vertical velocity of both atomic clouds at the beginning of the interferometer is set to $v_0 = 1.422(5)$ m/s.
Figure 5.1: Schematic picture of the $G$ measurement geometrical arrangement in configuration C1. Atoms and masses are shown from a top view ($xy$ plane), form a side view ($yz$ view) and in the time domain ($zt$ graph). $H_{\text{CL}}^\text{om}$ and $H_{\text{up}}^\text{om}$ are the heights of the lower and upper cylinders center of mass, whereas $h_{\text{om}}$ and $h_{\text{up}}$ are the ballistic turning points for the two atomic samples. $z_{\text{om}}^0$ and $z_{\text{up}}^0$ label the atomic vertical position at the time of the first interferometer pulse. The asymmetry of the interferometer in the time domain, introduced by the delay $\Delta t$ between the inversion time and the time of the central $\pi$ pulse, is also shown.

During the measurements the laboratory temperature was maintained within (20.0±0.1)$^\circ$C. This level of temperature stability was essential for a robust operation of the experimental apparatus.

5.2 Simulation of the experiment

The actual $G$ value must be extracted from the interferometric measurements by comparing the experimental differential phase shift to the result of a computer simulation that contains $G$ as the single parameter. The gravitational acceleration induced by all the moving masses during the experiment must be modelled with high accuracy. This acceleration is indeed a weak perturbation to the Earth gravity. Its maximum value is in fact of the order
of $10^{-7} \, g$ (see figure 5.4). The masses acceleration gradient is instead of the same order of magnitude of the Earth gradient, but the effect of the Earth gradient is cancelled because of the differential measurement scheme in the two configurations.

Only the masses that are moved between C1 and C2, i.e. cylinders and holding platforms, contribute to the final differential signal. We will now determine the gravitational potential and acceleration induced by a cylinder and by a torus. These will be used to evaluate the phase difference accumulated by the two atomic ensembles in the two masses configurations.

The simulations are still at a preliminary stage; single vertical position and velocity are taken into account to describe the motion of the atomic ensemble. A complete Monte Carlo simulation should be done for the accurate measurement, including a Gaussian distribution of the atoms in the 3D spatial dimensions and also in the velocity domain. The finite temperature of the cloud, inducing its spatial expansion in time, should be also considered, as well as the misalignment of the atomic fountain with respect to the mass distribution symmetry axis.

The preliminary simulation described in the following does not include all these features, but is enough to describe the experiment at the present accuracy level.

### 5.2.1 Gravitational potential and acceleration calculation:

**Cylinder**

Let us first determine the gravitational potential induced by a homogeneous cylinder of density $\rho$, radius $R$ and height $H$, placed in the origin of a cylindrical coordinate system $\hat{O}_{r\phi z}$.

#### Exact solution

The symmetry of the system suggests to use cylindrical coordinates. A point-like mass $m$ in $(r, z)$ experiences the potential

$$U(r, z) = Gm \rho \int_0^R \int_0^{2\pi} \int_0^H \frac{r' \, dz' \, d\phi' \, dr'}{\sqrt{(r' - r' \cos \phi')^2 + (r' \sin \phi')^2 + (z - z')^2}}$

$$= 2Gm \rho \int_0^R \int_0^{2\pi} \int_0^H \frac{r' \, dz' \, d\phi' \, dr'}{\sqrt{[(r + r')^2 + (z - z')^2][1 - \frac{4rr' \cos^2(\phi'/2)}{(r + r')^2 + (z - z')^2}]}}. \quad (5.1)$$
A Legendre elliptic integral of the first kind is defined as

\[ K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}. \]  

(5.2)

Changing the variable \( \phi = 2\psi \) in expression (5.1) we obtain

\[ U(r, z) = 2Gm \rho \int_0^R \int_{-\pi/2}^{\pi/2} \frac{4r'K(k) \, dz' \, dr'}{\sqrt{(r + r')^2 + (z - z')^2}} \]  

(5.3)

with \( k^2(r', z') = \frac{4r'z'}{(r + r')^2 + (z - z')^2} \). Noting that \( \frac{\partial}{\partial z} = -\frac{\partial}{\partial z'} \) we can evaluate the integral in \( dz' \) and obtain the following exact expression for the cylinder’s vertical acceleration on an axis parallel to the cylinder’s one

\[ a_z(r, z) = G \rho \int_0^R \left[ \frac{4r'K(k(r', \frac{H}{2})))}{\sqrt{(r + r')^2 + (z - \frac{H}{2})^2}} - \frac{4r'K(k(r', -\frac{H}{2})))}{\sqrt{(r + r')^2 + (z + \frac{H}{2})^2}} \right] \, dr'. \]  

(5.4)

This method provides the exact solution, but limited to the acceleration component along the cylinder’s axis direction. Furthermore elliptic integrals increase the evaluation time.

### Multipole expansion

An alternative method, computationally faster, consists in the determination of an approximated expression for the source masses potential. The potential of a cylinder is expanded in multipoles. In general the mass distribution contained within a sphere of radius \( R \) produces a gravitational potential outside the sphere that can be written as

\[ U(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} Y_{lm}(\theta, \phi) \frac{r^{l+1}}{r^{l+1}} \]  

(5.5)

with

\[ q_{lm} = -Gm \int_V Y_{lm}^*(\theta', \phi') r'(x') \, d\rho(x') \]  

(5.6)

Considering that the axial symmetry allows only \( m = 0 \) terms and that the symmetry for reflections on the horizontal plane passing through the center of mass lets all odd \( l \) terms vanish, (5.5) and (5.6) become

\[ U(r, \theta) = \sum_{l=0}^{\infty} \frac{q_{2l,0} P_{2l}(\cos \theta)}{r^{2l+1}} \]  

(5.7)

\[ q_{2l,0} = Gm \int_V P_{2l}(\cos \theta) r'(x') \, d\rho(x') \]  

(5.8)
where \( Y_{l0} = \sqrt{(2l + 1)/(4\pi)}P_l(\cos \theta) \) and \( P_l(\cos \theta) \) are the Legendre polynomials. Finally, the potential generated by our homogeneous cylindric mass distribution, with mass \( M \), height \( H \) and radius \( R \), can be described as

\[
U(r, \theta) = -\frac{GmM}{R} \sum_{l=0}^{\infty} \beta^{2l+1} Q_l(\alpha^2) P_{2l}(\cos \theta) \quad (5.9)
\]

where \( \alpha \) and \( \beta \) are the dimensionless coefficients expressed by

\[
\alpha = \frac{H}{2R} \quad \text{and} \quad \beta = \frac{R}{r} \quad (5.10)
\]

and \( Q_l(\alpha^2) \) are polynomials expressed by

\[
Q_l(\alpha^2) = \int_{-1}^{1} \int_{0}^{1} P_{2l}(\cos \theta')(v'^2 + (\alpha u')^2)v' \, du' \, dv'. \quad (5.11)
\]

with

\[
u' = \frac{2z'}{H} \quad v' = \frac{r'}{R} \quad \text{and} \quad \cos \theta' = \frac{\alpha u'}{\sqrt{v'^2 + (\alpha u')^2}} \quad (5.12)
\]

The polynomials explicitly depend only on \( \alpha \) because the two integrals of expression (5.11) can be analytically solved. From the potential it is possible to obtain an analytical expression also for the radial and axial component of the acceleration in the same approximation.

The smaller \( \alpha \) and \( \beta \), the faster the series (5.9) converges. In our case the closest cylinders lie at a distance \( r = 2R \) from the atoms, therefore \( \beta \) is fixed to 0.5, while \( \alpha \) is 1.5 when considering a whole cylinder and to reduce the relative error below \( 10^{-5} \) an expansion up to \( 2l = 40 \) is needed.

By dividing each cylinder in two halves the quantity \( R/r \) is reduced and the potential converges much faster since \( \alpha \) becomes 0.75. Truncation of the series at \( 2l = 10 \) is sufficient to compute the potential with at least 6 digits accuracy.

### 5.2.2 Gravitational potential and acceleration calculation: Torus

The platforms holding the masses can be decomposed into several tori with rectangular section. They must be taken into account in the vertical acceleration determination.

Let us consider a torus with internal radius \( R_{\text{in}} \), external one \( R_{\text{ext}} \) and height \( H \), lying on the \( xy \) plane symmetrically around the \( z \) axis.
The multipole expansion adopted for the cylinder description can not be used in this case because our probe mass is within the sphere that contains the source mass distribution. We choose then the expansion parameter

\[
\eta = \frac{r}{R_{\text{int}}}
\]  

(5.13)

that is expected to be much smaller than 1. The gravitational potential induced by the torus in \((r, z)\) is

\[
U(r, z) = \int_{0}^{H} \int_{R_{\text{int}}}^{R_{\text{ext}}} \int_{0}^{2\pi} \frac{Gm \rho \, dr' \, d\phi' \, dz'}{\sqrt{(z' - z)^2 + (r')^2 + r^2 - 2rr' \cos \phi'}}
\]  

(5.14)

We introduce the following dimensionless variables

\[
u' = \frac{r'}{R_{\text{int}}} \quad \text{and} \quad v' = \frac{z'}{R_{\text{int}}}
\]  

(5.15)

and the parameters

\[
\alpha = \frac{R_{\text{ext}}}{R_{\text{int}}} \quad \text{and} \quad \beta = \frac{H}{R_{\text{int}}}
\]  

(5.16)

and get to

\[
U(\eta, v) = Gm \rho R_{\text{int}}^2 \int_{0}^{\beta} \int_{1}^{\alpha} \int_{0}^{2\pi} \frac{u' \, dv' \, du' \, dv'}{\sqrt{(v' - v)^2 + (u')^2 + \eta^2 - 2u'\eta \cos \phi'}}
\]  

(5.17)

The integrand can be expanded in series up to the second order in \(\eta\)

\[
\frac{u'}{[(v' - v)^2 + (u')^2]^{1/2}} + \frac{u' \eta \cos \phi'}{[(v' - v)^2 + (u')^2]^{3/2}} + \frac{2u' \eta^2}{3\eta^2 (u')^2 \cos^2 \phi'} + \frac{2[(v' - v)^2 + (u')^2]^{3/2}}{2[(v' - v)^2 + (u')^2]^{5/2}}
\]

leading, after integrating in \(d\phi'\) and \(du'\), to

\[
U(\eta, v) = Gm \rho R_{\text{int}}^2 \int_{0}^{\beta} I(\eta, v) \, dv
\]  

(5.18)

with

\[
I(\eta, v) = 2\pi \left( \sqrt{\alpha^2 + (v' - v)^2} - \sqrt{1 + (v' - v)^2} \right) dv' + \frac{\pi \eta^2}{2} \left( \frac{1}{[1 + (v' - v)^2]^{3/2}} - \frac{\alpha^2}{[\alpha^2 + (v' - v)^2]^{3/2}} \right) dv'.
\]
This integral can be analytically solved, but we are interested in the acceleration expression. Its vertical component is easily deduced

\[ a_z(\eta, v) = G m \rho R \pi [\text{sign}(v - \beta) I(\eta, \beta) - \text{sign}(v) I(\eta, 0)]. \] (5.19)

An expression for the radial component of the acceleration can be determined as well by applying the Gauss theorem. In cylindrical coordinates

\[ \nabla \cdot \mathbf{a} = 0 \quad \Rightarrow \quad \frac{\partial a_z}{\partial z} = -\frac{1}{r} \frac{\partial}{\partial r}(r a_r). \] (5.20)

The first term in \( \eta \) different from zero is

\[ a_r(v, \eta) = G m \rho R \pi \eta \left( \frac{\beta - v}{\sqrt{1 + (\beta - v)^2}} - \frac{\beta - v}{\sqrt{\alpha^2 + (\beta - v)^2}} + \right. \]
\[ + \left. \frac{v}{\sqrt{1 + v^2}} - \frac{v}{\sqrt{\alpha^2 + v^2}} \right). \] (5.21)

The potential and the acceleration components along the symmetry axis can be easily determined by choosing \( \eta = 0 \).

### 5.2.3 Determination of the interferometer phase

Two different approaches have been used to evaluate the gravitationally induced phase shift. For at most quadratic potentials with respect to \( x \) and \( v \) an exact full quantum mechanical phase calculation method can be found in [92], but this correction is not limiting the accuracy at the present level.

The first method consists in the determination of the atomic trajectory \( z(t) \) considering the exact expression for the overall acceleration of the 24 cylinders and the 2 platforms, in the two experimental arrangements, added to Earth gravity \( g \) with its linear gradient \( \gamma \) over 30 cm distance. The motion was considered vertical, along the central symmetry axis. The equations of motion for the two atomic samples during the interferometer sequence are solved with a numerical integration based on a fourth-order Runge–Kutta algorithm [93].

The second method relies on a perturbative approach. We start from an unperturbed system with only uniform gravity \( g \). Its potential \( V(z) = mgz \) determines the unperturbed atomic trajectory \( \tilde{z}(t) \). By adding a perturbative potential \( U(z) \), containing the effect of the source masses and that of the Earth gravity gradient, we want to determine the new trajectory \( z(t) = \tilde{z}(t) + \delta z \), therefore to estimate \( \delta z \). If the total energy of the system is

\[ E = \frac{1}{2} m v(z)^2 + V(z) + U(z) \] (5.22)
we have
\[
\frac{1}{v(z)} = \frac{1}{\sqrt{\frac{2}{m}[E - V(z) - U(z)]}}.
\]

An integration in \(dz\) yields to an expression for \(t(z)\)
\[
t(z) = \int_{z_0}^{z} \frac{dz'}{v(z')} = \int_{z_0}^{\bar{z} + \delta z} \frac{dz'}{\sqrt{\frac{2}{m}[E - V(z') - U(z')]}},
\]

\[= F(\bar{z} + \delta z). \quad (5.24)\]

We need an expression for \(\delta z\) at the times of the interferometer pulses
and the idea to obtain it is the following
\[
t = F(\bar{z} + \delta z)
\]
\[
t = F(\bar{z}) + \delta z F'(\bar{z}) = F(\bar{z}) + \delta z \frac{1}{v(\bar{z})}
\]
\[
\delta z = v(\bar{z})[t - F(\bar{z})].
\]

\([t - F(\bar{z})]\) is a small difference between two large quantities. Since we are
not interested in estimating the two quantities themselves, but only in their
difference, this procedure allows us to directly determine \([t - F(\bar{z})]\) reducing
numerical stability problems.

The accumulated phase shift in one interferometer because of \(U(z)\) is
then simply estimated by
\[
\Delta \phi = k_{\text{eff}} [\delta z(2T) - 2\delta z(T)] \quad (5.25)
\]

Far enough from the inversion point of the atomic trajectory \(x = U(z)/[E -
V(z)] \ll 1\) and the integrand can be expanded in series of \(x\). In the small
region where this is no longer valid we can approximate the perturbative
potential with a first order expansion in Taylor series around the inversion
point of the unperturbed trajectory \(\bar{z}_i\)
\[
U(z) = U(\bar{z}_i) + (z - \bar{z}_i)U'(\bar{z}_i).
\]

\((5.26)\) one can find an expression for \(\delta z\) in the relative interval, it
is then connected to the \(\delta z\) found for the trajectory far from the inversion
point.

The validity of this approximation method has been checked by compar-
ison with the analytical solution known in case of the gravity gradient only.
The calculated phase shift resulted consistent with the expected one within
\(10^{-6}\).
5.3 Experimental results with lead cylinders

While the final INERMET cylinders were being machined and tested a first demonstration of mass detection was made using lead cylinders that have similar geometrical dimensions, but different density and surface roughness. Their average mass is 12.863 kg with a standard deviation of 22 g for the 24 samples, yielding a density of 10870(20) kg/m³.

The vertical positions of the two sets of masses and those of the atomic samples, are reported below

<table>
<thead>
<tr>
<th>$H_{up}^{C2}$ = 755 mm</th>
<th>$h_{up}$ = 647.0 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{up}^{C1}$ = 555 mm</td>
<td>$z_{up}$ = 525.1 mm</td>
</tr>
<tr>
<td>$H_{dw}^{C1}$ = 355 mm</td>
<td>$h_{dw}$ = 272.7 mm</td>
</tr>
<tr>
<td>$H_{dw}^{C2}$ = 155 mm</td>
<td>$z_{dw}$ = 150.8 mm</td>
</tr>
</tbody>
</table>

A long run of about 11 hours of gradiometric measurements was first performed, without actively changing any experimental parameter. An indication of the system stability can be obtained from the Allan variance of the differential phase in one configuration.

Consider $N$ different measurements $y_i$ of a quantity $y$ that are equally delayed by a time $\tau$. By grouping the results into data sets of $m$ consecutive measurements we obtain $k = N/m$ intervals, each with average value $\langle y \rangle_k$. The Allan variance [94] on $y$ for a time $t_m = m \cdot \tau$ is defined as

$$
\sigma_y^2(t_m) = \lim_{N \to \infty} \frac{1}{2k} \sum_{k=1}^{N-1} (\langle y \rangle_{k+1} - \langle y \rangle_k)^2.
$$

Figure 5.2 shows the Allan variance relative to the acquired data. The characteristic slope of $\sigma_{\Delta \phi}(t_m) \propto 1/t_m$ continues without flattening. Up to 40000 s the error in the determination of the relative phase scales down proportionally to the square root of the number of measurements as gaussian-like distributed data do. No $1/f$ noise emerges in these measurement times.

For both the C1 and C2 configurations, 144 gradiometric measurements were performed. The Raman phase was varied between the central $\pi$ pulse and the last $\pi/2$ from 0° to 355° in 5° steps, for two times. For each phase step the normalized population of the F=1 state in both clouds was detected and the data were plotted, fitted to an ellipse and finally the resulting differential phase represented in figure 5.3. The second and the third short coils
were turned on for 10 ms while the lower cloud was nearby, introducing a differential phase shift due to the second order Zeeman effect. The pulsed magnetic field added a phase shift that was detected in the same way also in the other measurement configuration. Since the magnetic pulse area is stable to better than 0.1%, it is not a limit for the measurement accuracy. Its stability was checked by measuring the fluctuation of the magnetic pulse area (temporal length × field amplitude). Each single measurement lasted 5.76 s, basically because of the first MOT loading and the time of flight of the clouds, therefore 830 s were needed to obtain each point shown in the lower graph of figure 5.3. The masses were then moved to configuration C2. This operation required about 150 s including the time necessary to damp mechanical oscillations induced by the movement. The same procedure was adopted and a different elliptic distribution was obtained. The new phase difference resulting from the fit is reported as well in figure 5.3. The elliptic fit over 144 measurements determined the relative phase with an error of $\sigma_{\phi} = 14$ mrad, therefore the sensitivity was 400 mrad/√Hz. The differential phase between C1 and C2 had a statistical uncertainty of $\sigma_{\Delta \phi} = 20$ mrad.

The measurement cycle was repeated 11 times without any interruption while the temperature was kept within $(20.0\pm0.1)^\circ C$. The resulting differential phase shift was

$$\Delta \Phi = \Delta \phi^{C1} - \Delta \phi^{C2} = 395(4) \text{ mrad}$$ \hspace{1cm} (5.28)

The phase value was determined by considering $\Delta \phi^{C2}$ and comparing it
5.3 Experimental results with lead cylinders

Figure 5.3: Measured phase difference between two simultaneous, vertically displaced atom interferometers. Red points were obtained with lead source masses close to each other (C1) and blue points by moving them far apart (C2). The two atomic clouds launch heights were the same in both configurations. See text for actual positions. Each point is obtained by fitting ellipses as those shown in the two insets.

to the average of the previous $\Delta \phi^{C1}$ and the following one, in order to approximate drifts with a linear term.

The values resulting from the elliptic fit were corrected taking into account the amplitude noise level (about 5%) and the bias induced by the ellipse fitting method as a function of the phase (see figure 4.15).

Including the geometrical distribution of masses (lead cylinders and holding platforms) in the computer simulations a first value of $G$ was obtained

$$G = 6.67(6) \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$

in excellent agreement with the CODATA value.
5.4 Experimental results with tungsten cylinders

The $G$ measurement performed with the lead cylinders and reported in the previous section has a large uncertainty, but is consistent with the last CODATA recommended value. This encourages to work further on the experimental apparatus to improve its stability and reduce the statistical error.

If the statistical uncertainty is reduced, the experimental parameters and the way they influence the determination of $G$ have to be known with a better resolution in order to control the systematics. Atoms--masses relative distances are crucial parameters in our a measurement. Since it is technically much easier with our apparatus to measure the source masses position in space and their relative distance in the two configurations, than the atomic positions, we decided to push towards a highly accurate localization of the masses and to place the atoms where the sensitivity to their position is at minimum. We need to find a particular parameters choice that allows the overall vertical acceleration to have two flat regions, i.e. where the Earth gravity gradient is compensated by the source masses, vertically displaced by about 30 cm for both masses configurations. For the masses geometry we chose this can not happen with lead cylinders. Tungsten cylinders, instead,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5_4.png}
\caption{Simulated vertical acceleration induced by the tungsten cylinders along the central axis of the system in C1 (red) and C2 (blue). A linear slope taking into account the gravity gradient is added to the acceleration. The vertical regions where the two interferometers are realized are shown. It is clear that the high density of the source masses allows a gradient compensation and flat regions for both configurations. This does not happen with lead cylinders.}
\end{figure}
are dense enough to allow the desired stationary zones (see figure 5.4).

In the following section the procedure to determine the best parameters choice is outlined.

### 5.4.1 Best parameters configuration

Once $T$ and $v_0$ are chosen, the determination of the best parameters configuration is done in the following way:

In configuration C1 the two sets of masses are put as close as possible ($H_{dw}^{C1}$ and $H_{up}^{C1}$ are set). This is the way to create the maximum differential vertical acceleration (see figure 5.4).

Using these input parameters the experiment is simulated for different values of the vertical atomic positions $z_{dw0}$ and $z_{up0}$ at the beginning of the interferometer sequence. As reported in figure 5.5 the differential phase shift $\Delta \Phi^{C1}$ has a stationary point that satisfies our request of minimum dependency on the atomic position. The corresponding values $Z_{dw0}^{C1}$ and $Z_{up0}^{C1}$ are chosen for the experiment.

![Figure 5.5: Simulation of the differential phase shift $\Delta \Phi^{C1}$ (rad) accumulated between the two atomic samples with tungsten masses in configuration C1 as a function of the initial position of the two atomic samples $z_{dw0}$ and $z_{up0}$. The chosen value $Z_{dw0}^{C1}$ and $Z_{up0}^{C1}$ are the ones for which phase $\Delta \Phi^{C1}$ is stationary.](image-url)
Figure 5.6: (Above) Simulation of the differential phase shift $\Delta \phi$ obtained with atomic initial positions fixed and set to $Z_{dw}^0$ and $Z_{up}^0$, as a function of the two sets of masses vertical positions $H_{dw}$ and $H_{up}$. (Below) Modulus of the phase gradient with respect to $z_{dw}^0$ and $z_{up}^0$, evaluated for $Z_{dw}^0$ and $Z_{up}^0$ as a function of $H_{dw}$ and $H_{up}$. Comparing the two graphs one can see that the gradient does not vanish for the same $H$ values for which the phase is stationary. Among the four zero points we choose the one with the maximum phase value for the CO masses configuration.
The next step consists in the determination of the second configuration of masses, labelled with \( H_{\text{up}}^{C_2} \) and \( H_{\text{dw}}^{C_2} \), provided the atomic parameters are kept unchanged. Again what we want to minimize is the sensitivity of the phase shift to the atomic positions, therefore we choose the new masses positions corresponding to the vanishing of the phase gradient with respect to the atomic initial positions. Considering the multifunction \( \Delta \phi(z^{\text{dw}}_0, z^{\text{up}}_0, H_{\text{dw}}, H_{\text{up}}) \) we want

\[
\begin{align*}
\left. \frac{\partial}{\partial z^{\text{dw}}_0} \Delta \phi(z^{\text{dw}}_0, z^{\text{up}}_0, H_{\text{dw}}, H_{\text{up}}) \right|_{z^{\text{dw}}_0, z^{\text{up}}_0} &= 0 \\
\left. \frac{\partial}{\partial z^{\text{up}}_0} \Delta \phi(z^{\text{dw}}_0, z^{\text{up}}_0, H_{\text{dw}}, H_{\text{up}}) \right|_{z^{\text{dw}}_0, z^{\text{up}}_0} &= 0
\end{align*}
\]

(5.29)

The phase \( \Delta \phi \) that is obtained with \( z^{\text{dw}}_0 = Z^{\text{dw}}_0 \) and \( z^{\text{up}}_0 = Z^{\text{up}}_0 \) is plotted as a function of the masses positions in figure 5.6 and it is compared with the modulus of the phase gradient with respect to \( z^{\text{dw}}_0 \) and \( z^{\text{up}}_0 \) evaluated again in \( z^{\text{dw}}_0 = Z^{\text{dw}}_0 \) and \( z^{\text{up}}_0 = Z^{\text{up}}_0 \) and as a function of \( H_{\text{dw}} \) and \( H_{\text{up}} \).

It is clear that the gradient does not vanish for the same \( H \) values for which the phase is maximized or minimized. If we perform the measurement with \( H_{\text{dw}} \) and \( H_{\text{up}} \) that maximize the phase we would get a large signal, but at the same time we would be more sensitive to fluctuations in the atomic positions. A change of 0.5 mm in \( z_0 \) would imply a variation on \( G \) of \( 2 \times 10^{-3} \).

We perform the measurement by choosing \( H_{\text{dw}} \) and \( H_{\text{up}} \) for which the gradient is zero. With this choice a 0.5 mm error in \( z_0 \) leads to a systematic error on \( G \) of \( 1.2 \times 10^{-5} \), whereas the phase is only slightly reduced by 20%.

Among the four zero–gradient points shown in figure 5.6 the one on the bottom right corresponds to configuration C1, set from the beginning. Configuration C2 is chosen in order to have \( \Delta \phi^{C_2} - \Delta \phi^{C_1} \) as large as possible, therefore the upper stationary point to the left.

### 5.4.2 Measurement

Following the described procedure, the best parameter were obtained and are reported in the table below

| \( H_{\text{up}}^{C_2} \) = 672.88 mm | \( h_{\text{up}}^{C_2} \) = 604.9 mm |
| \( H_{\text{up}}^{C_1} \) = 555.40 mm | \( z_0^{\text{up}} \) = 501.8 mm |
| \( H_{\text{dw}}^{C_1} \) = 355.40 mm | \( h_{\text{dw}}^{C_1} \) = 296.1 mm |
| \( H_{\text{dw}}^{C_2} \) = 238.45 mm | \( z_0^{\text{dw}} \) = 193.0 mm |

The first preliminary measurement of \( G \) performed with tungsten cylinders in the "best parameter configuration" consisted in a sequence of 2000
gradiometric measurements in C1 and then 2500 in C2. A bias magnetic field of 250 mG was present and the short coils (number 2, 3 and 4) were turned on generating a 5 ms pulsed magnetic field on the lower cloud during the second part of the interferometer. With 20 points it was possible to obtain an elliptic distribution and to determine its phase \( \phi \) with a minimum uncertainty \( \sigma_\phi \sim 15 \text{ mrad} \). Each measurement lasted 7 s, therefore a phase determination of 200 mrad/\( \sqrt{\text{Hz}} \) was possible\(^1\), improving of about a factor 2 with respect to the previous measurement with lead cylinders. The results are reported in figure 5.7.

\(^1\)For the preliminary measurements reported here 200 mrad/\( \sqrt{\text{Hz}} \) is the best phase determination obtained and not the average value.
Averaging over all the obtained phase values in the two configurations we have

\[ \Delta \phi^{C1} = 1.001(4) \text{ rad} \quad \text{and} \quad \Delta \phi^{C2} = 503(3) \text{ mrad}. \quad (5.30) \]

As done for the measurement with lead, these values were corrected considering the bias introduced by the elliptic fit yielding

\[ \Delta \phi^{C1} = 999(4) \text{ mrad} \quad \text{and} \quad \Delta \phi^{C2} = 496(3) \text{ mrad}. \quad (5.31) \]

For the final measurement the magnetic field pulse, adding the constant relative phase between upper and lower interferometer, will be chosen in order to have two ellipses symmetrically located relative to the \( \pi/2 \) differential phase. In this way the bias induced by the ellipse fitting method would be common-mode and therefore minimized in the differential phase extraction.

The measured differential gravitationally induced shift is

\[ \Delta \Phi = \Delta \phi^{C1} - \Delta \phi^{C2} = 503(5) \text{ mrad} \quad (5.32) \]

The value for \( G \) extracted from the simulation in this case is

\[ G = 6.61(7) \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \]

where the indicated uncertainty is only statistical. As reported in table 5.1, a quadrature sum over all the present systematic errors estimated yields \( 9 \times 10^{-4} G \). Launch direction in the two configurations is not included in the budget because not investigated yet.

### 5.4.3 Error analysis

Sources of the measured uncertainty and of possible systematic errors on the final value of \( G \) are discussed here. An error budget is also reported in order to summarize all those contributions and to present a clear picture of what is presently limiting the knowledge of \( G \) in this experiment.

The uncertainties related to the experimental parameters are listed in table 5.1. For each parameter its magnitude, the present and the projected uncertainties are reported.

Using the simulation, the effect of each uncertainty on the final \( G \) value was determined and is reported as well. The results showed the sensitivity of the \( G \) determination to the parameters knowledge.

For these calculations the axis of the cylinders were assumed to be parallel to the vertical direction. Being physically in contact, their radial positions
were assumed to be known with the same error as that of their radius and that of the inner ring; a factor 2 was added for the external cylinders. The vertical position, relative to the reference plane in C1 was measured with a surface gauge with an accuracy of 50 µm and a resolution of 10 µm. The positions in C2 (relative to C1) were measured with the optical rulers (1 µm resolution and 10 µm accuracy). The moving platform dimensions are derived from the blueprints. Machining accuracy is claimed to be 10 µm.

Nominal densities for the platform materials were used to derive masses since presently direct measurements can not be performed.

Each cylinder was weighted with a calibrated precision balance (2 g sensitivity). Diameters and heights have been measured by the manufacturer. Regarding the sensitivity on the density homogeneity of Inermet cylinders, an upper estimate was done considering the largest density difference $\Delta \rho_{\text{max}}$ measured on the small blocks (see density test results in section 3.3.1) out of the test cylinder.

The effect of two volumes of 28 cm$^3$, with opposite sign density variation $\pm \Delta \rho_{\text{max}}/2$, placed at the top and at the bottom of a cylinder, close to the atomic trajectory has been evaluated. The result was multiplied by a factor $\sim \sqrt{2\pi}$ considering the random distribution over the whole cylinder volume$^2$. This was separately done for internal (int) and external (ext), upper (up) and lower (dw) cylinders in groups of 6. The results are reported in table 5.1.

Atomic initial velocity and positions were obtained by time of flight measurements of the atoms passing twice through a horizontal sheet of light, whose position relative to the reference plane was measured within 100 µm with a caliper. Launch direction measurements have not been performed yet, but considering the launch heights and the juggling recapture parameters we can put an upper estimate to the launch direction tilt with respect to the local vertical direction of 2 mrad. If the tilt does not change from C1 to C2 the Coriolis effect does not enter the measurement, it is common mode and therefore cancelled out. We need to avoid that the launch direction somehow changes from C1 to C2, in a correlated way. A crucial parameter that needs to be investigated is, for example, the magnetic field change in the trap chamber induced by the masses movement. A magnetic material in fact can modify the external magnetic field in which it is immersed and if this change is present in the trap region the launch direction can be affected. This is the

$^2$The entire cylinder volume corresponds to about 42 times the volume of a single 28 cm$^3$ block.
reason why we chose non-magnetic materials for realizing source masses and their support. By the way a residual magnetic susceptibility could provoke a relevant effect and a characterization is therefore needed.

The magnetic field in the vacuum tube is dominated by the stable one generated by the bias coil. External fields are attenuated by the \( \mu \)-metal shields by 69 dB in the axial directions and by 76 dB in the radial one. If the external field is varied by 1 G moving the masses from C1 to C2, then the variation in the interferometer region would be lower than 100 nG. As explained in section 2.3 only magnetic fields that are different around the two atomic trajectories and that change between the first and the second part of the interferometer in a different way for the two masses configurations are detected as systematic effect, therefore this is not an issue in this experiment.
<table>
<thead>
<tr>
<th>SOURCE</th>
<th>Magnitude</th>
<th>Present uncert.</th>
<th>(\Delta G/G) (10(^{-4}))</th>
<th>Aimed uncert.</th>
<th>(\Delta G/G) (10(^{-4}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Int. cyl. radial position</td>
<td>100.00 mm</td>
<td>50 (\mu m)</td>
<td>1.1</td>
<td>10 (\mu m)</td>
<td>0.22</td>
</tr>
<tr>
<td>Ext. cyl. radial position</td>
<td>173.00 mm</td>
<td>100 (\mu m)</td>
<td>0.5</td>
<td>10 (\mu m)</td>
<td>0.05</td>
</tr>
<tr>
<td>(H_{up}^{C1} - H_{dw}^{C1})</td>
<td>200.00 mm</td>
<td>100 (\mu m)</td>
<td>3.8</td>
<td>10 (\mu m)</td>
<td>0.38</td>
</tr>
<tr>
<td>(H_{up}^{C2} - H_{dw}^{C2})</td>
<td>434.43 mm</td>
<td>100 (\mu m)</td>
<td>3.0</td>
<td>10 (\mu m)</td>
<td>0.30</td>
</tr>
<tr>
<td>Cylinder mass</td>
<td>21552 g</td>
<td>3 g</td>
<td>1.4</td>
<td>1 g</td>
<td>0.50</td>
</tr>
<tr>
<td>Dens. homog. up-int</td>
<td>18249 kg/m(^3)</td>
<td>24 kg/m(^3)</td>
<td>0.21</td>
<td>24 kg/m(^3)</td>
<td>0.21</td>
</tr>
<tr>
<td>Dens. homog. dw-int</td>
<td>18249 kg/m(^3)</td>
<td>24 kg/m(^3)</td>
<td>0.01</td>
<td>24 kg/m(^3)</td>
<td>0.01</td>
</tr>
<tr>
<td>Dens. homog. up-ext</td>
<td>18249 kg/m(^3)</td>
<td>24 kg/m(^3)</td>
<td>0.03</td>
<td>24 kg/m(^3)</td>
<td>0.03</td>
</tr>
<tr>
<td>Dens. homog. dw-ext</td>
<td>18249 kg/m(^3)</td>
<td>24 kg/m(^3)</td>
<td>(\ll 1)</td>
<td>24 kg/m(^3)</td>
<td>(\ll 1)</td>
</tr>
<tr>
<td>Support platform mass</td>
<td>24930 g</td>
<td>60 g</td>
<td>0.8</td>
<td>1 g</td>
<td>(\ll 1)</td>
</tr>
<tr>
<td>Support ring mass</td>
<td>337.7 g</td>
<td>0.5 g</td>
<td>0.04</td>
<td>0.5 g</td>
<td>(\ll 1)</td>
</tr>
<tr>
<td>(\xi_{up}^{0})</td>
<td>501.8 mm</td>
<td>0.5 mm</td>
<td>0.13</td>
<td>0.1 mm</td>
<td>0.02</td>
</tr>
<tr>
<td>(\xi_{dw}^{0})</td>
<td>193.0 mm</td>
<td>0.5 mm</td>
<td>0.12</td>
<td>0.01 mm</td>
<td>0.02</td>
</tr>
<tr>
<td>(v_0)</td>
<td>1.422 m/s</td>
<td>5 mm/s</td>
<td>2.3</td>
<td>1 mm/s</td>
<td>0.05</td>
</tr>
<tr>
<td>Launch direction C1/C2</td>
<td>(&lt; 2) mrad</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Table tilt C1/C2</td>
<td>10 (\mu rad)</td>
<td>10 (\mu rad)</td>
<td>6.4</td>
<td>1 (\mu rad)</td>
<td>0.64</td>
</tr>
<tr>
<td>Earth gradient (\gamma)</td>
<td>(3.08 \cdot 10^{-6}/s^2)</td>
<td>(3 \cdot 10^{-8}/s^2)</td>
<td>(1)</td>
<td>(1 \cdot 10^{-8}/s^2)</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td></td>
<td></td>
<td>8.7</td>
<td></td>
<td>0.96</td>
</tr>
</tbody>
</table>

Table 5.1: Present and projected error budget of the \(G\) measurement.
Chapter 6

Conclusions

6.1 Results

This PhD work had as main goal the development of an atomic gravity-gradiometer for a new accurate determination of the gravitational constant $G$.

During the first year the optimization of the laser system brought to a stable atomic fountain. Clouds of up to $5 \times 10^9$ atoms with a minimum temperature of 4 μK could be launched up to 1.25 m, only limited by the apparatus dimensions. The first Raman atom interferometer was realized on the freely falling atoms, providing a sensitivity of $10^{-5} \, g/\sqrt{\text{Hz}}$.

The second year was dedicated to the juggling method for the launch in a rapid sequence of two samples of $5 \times 10^8$ cold atoms. That allowed to realize the first gradiometric measurement and differential signals on samples 30-40 cm distant were detected, with a sensitivity of $6 \times 10^{-8} \, g/\sqrt{\text{Hz}}$ on the differential acceleration. In the mean time some effort was put on the study and characterization of the tungsten source masses and on the calibration of their holder and elevator.

In the third year the whole experimental system was characterized and the computer simulation was optimized and checked in order to perform the $G$ measurement. At the beginning lead cylinders were used and a demonstration of the mass detection with the gradiometer was done. Then at the end of the year, the lead cylinders were replaced with the denser tungsten ones and a measurement was performed. Preliminary results provided a determination of $G$ at the 1% level. The doubly differential measurement showed to cancel out most of the noise and of the systematics and a long term stability
in the measurement of $G$ was checked observing the Allan variance.

Further improvements on the experimental apparatus have to be done to reduce the statistical uncertainty. In particular one should enhance the number of atoms contributing to the signal; an optimization of the launch sequence parameters to further cool the atoms and a more efficient method of selecting the state and the velocity after the launch could easily allow to exceed $10^6$ atoms contributing to the signal. Then a stabilization of the laser beam intensity and the realization of a less noisy detection system would increase the S/N to the quantum projection noise limit of 1000/1.

Regarding systematics one should accurately detect the cylinders positions with a laser tracker to below 10 $\mu$m during the measurements and characterize the possible launch direction changes correlated with the two masses configurations with resolutions better than 1 $\mu$rad. A check of the launch heights should be included in the experimental sequence in order not to add systematic shifts in the measurement due to a not accurate knowledge of the atomic position at the time of the interferometer pulses.

With these further improvements on the apparatus it will be eventually possible to determine the gravitational constant to better than 100 ppm, contributing with this cold-atom interferometry method to the accuracy of $G$ value.

### 6.2 Future prospects

Experiments like the one presented in this thesis could basically have two different development ways. On the one hand one can focus on the realization of transportable devices \cite{95} and on the other hand use the advantages of the atomic sensors to implement other precise measurements of fundamental physics.

Transportable highly sensitive atom gravimeters, gradiometers or gyroscopes would be appreciated in Earth science applications like aquifer and mine monitoring, mineral exploration, magma migration in volcanos systems and earthquakes monitoring. Geophysicists are looking at the evolution of atomic sensors with a great interest because absolute measurements with high accuracy would be then possible using transportable devices. They are independent absolute devices and do not need calibrations. Spring mass relative gravimeters are commonly used as survey instruments \cite{96}, but are not absolute and need to be calibrated on an absolute one that is usually located in fixed stations. Superconducting gravimeters are very sensitive, but
not absolute neither transportable, whereas free falling corner cubes–based optical interferometers are characterized by a high accuracy, but can not be transported and are very expensive. Atom–based devices could replace present techniques used for orientation in navigation and could also be installed on space vehicles with different purposes, like for example to perform precision measurement in a microgravity, high vacuum and vibration free environment [97].

Building new transportable devices means working on the volume occupation reduction of the whole vacuum, optical and electronic system, to develop robust laser locks and care about vibration suppression techniques. The realization of traps in microchips could be useful for the compactness purpose.

The other research line is more related with fundamental physics and precision measurements. The high sensitivity of the atoms could allow to detect very small quantities with relevant accuracy. Among the possible developments of this experiment, after the final measurement of $G$, there is the test of matter neutrality [60]. The fact that electron and proton charge compensate providing neutral atoms is tested to a level of about $10^{-22}$. Atom interferometers could surpass that level by orders of magnitude exploring smaller ranges of charge imbalance. A great interest is also preserved for gravity measurements at very short distances, on the micrometer distance scale. Some preliminary measurements are already running with the purpose to verify whether at such short distances the Newtonian gravitational law is still valid or if new physics is present. In principle one could also develop a large area atom interferometer for the detection of gravitational waves [98].
Appendix A

$^{87}$Rb physical properties

Some $^{87}$Rb typical quantities that can be useful while reading this thesis are reported in the two following tables. Table A.1 contains general physical properties of $^{87}$Rb, whereas in table A.2 more specific information regarding the $D_2$ transition in particular are reported. The data are taken from [99].

<table>
<thead>
<tr>
<th>Atomic Number</th>
<th>$Z$</th>
<th>37</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total nucleons</td>
<td>$Z + N$</td>
<td>87</td>
</tr>
<tr>
<td>Nuclear spin</td>
<td>$I$</td>
<td>3/2</td>
</tr>
<tr>
<td>Relative natural abundance</td>
<td>$\eta$($^{87}$Rb)</td>
<td>27.83(2) %</td>
</tr>
<tr>
<td>Nuclear lifetime</td>
<td>$\tau_n$</td>
<td>$4.88 \times 10^{10}$ yr</td>
</tr>
<tr>
<td>Vapor pressure $@$ 25°C</td>
<td>$P_v$</td>
<td>$3.0 \times 10^{-7}$ torr</td>
</tr>
<tr>
<td>Atomic mass</td>
<td>$m$</td>
<td>1.44316060(11) $\times 10^{-25}$ kg</td>
</tr>
<tr>
<td>Ground state hyperf. splitting</td>
<td>$\nu_{ab}$</td>
<td>6.834682610 90429(9) GHz</td>
</tr>
<tr>
<td>$D_2$ dipole matrix element</td>
<td>$\langle J=\frac{1}{2}</td>
<td>m</td>
</tr>
<tr>
<td>$D_1$ dipole matrix element</td>
<td>$\langle J=\frac{1}{2}</td>
<td>m</td>
</tr>
</tbody>
</table>

Table A.1: Relevant data of $^{87}$Rb.

A scheme including the relative strength of all possible transition between each hyperfine magnetic sublevel of the ground state $5^2 S_{1/2}$ and the excited state $5^2 P_{3/2}$, useful for the determination of the allowed Raman transitions, and the energy levels of rubidium are reported as well.
<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>$\nu_{2-3}$</td>
</tr>
<tr>
<td>Transition energy</td>
<td>$h\nu_{2-3}$</td>
</tr>
<tr>
<td>Wavelength (Vacuum)</td>
<td>$\lambda_{\text{vac}}$</td>
</tr>
<tr>
<td>Wavelength (Air)</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Wavevector (Vacuum)</td>
<td>$k$</td>
</tr>
<tr>
<td>Lifetime</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Natural linewidth (FWHM)</td>
<td>$\Gamma$</td>
</tr>
<tr>
<td>Saturation intensity</td>
<td>$I_S(\sigma)$</td>
</tr>
<tr>
<td>Recoil velocity</td>
<td>$v_r$</td>
</tr>
<tr>
<td>Recoil temperature</td>
<td>$T_r$</td>
</tr>
<tr>
<td>Recoil frequency</td>
<td>$\nu_r$</td>
</tr>
<tr>
<td>Recoil energy</td>
<td>$E_r$</td>
</tr>
<tr>
<td>Doppler temperature</td>
<td>$T_D$</td>
</tr>
<tr>
<td>II Zeeman shift ($m_r=0$)</td>
<td>$a_{Z,II}$</td>
</tr>
</tbody>
</table>

Table A.2: Specific rubidium data relative to the D$_2$ transition of $^{87}\text{Rb}$. 

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$384.2281152033(77)$ THz</td>
</tr>
<tr>
<td></td>
<td>$1.589049439(58)$ eV</td>
</tr>
<tr>
<td></td>
<td>$2.5459376 \times 10^{-19}$ J</td>
</tr>
<tr>
<td></td>
<td>$780.241209686(13)$ nm</td>
</tr>
<tr>
<td></td>
<td>$780.03200$ nm</td>
</tr>
<tr>
<td></td>
<td>$80528.75481555$ m$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$26.24(4)$ ns</td>
</tr>
<tr>
<td></td>
<td>$2\pi \cdot 6.065(9)$ MHz</td>
</tr>
<tr>
<td></td>
<td>$1.67$ mW/cm$^2$</td>
</tr>
<tr>
<td></td>
<td>$5.8845$ mm/s</td>
</tr>
<tr>
<td></td>
<td>$361.96$ nK</td>
</tr>
<tr>
<td></td>
<td>$3.7710$ kHz</td>
</tr>
<tr>
<td></td>
<td>$2.499 \times 10^{-30}$ J</td>
</tr>
<tr>
<td></td>
<td>$1.56 \times 10^{-11}$ eV</td>
</tr>
<tr>
<td></td>
<td>$146$ $\mu$K</td>
</tr>
<tr>
<td></td>
<td>$575.15$ Hz/G$^2$</td>
</tr>
</tbody>
</table>
Figure A.1: The two ground state and the four hyperfine levels of $5^2 P_{3/2}$ together with all the relative magnetic sublevels are indicated. The number in the circles indicate the transition strength multiplied by 120. $\pm N$ stands for a Clebsch–Gordan coefficient equal to $\pm \sqrt{N/120}$. 
Figure A.2: Rubidium $D_2$ transitions from the upper hyperfine level of the ground state. Indicated are the transition frequencies, information about the levels and below the corresponding saturated absorption spectroscopy signal. The indications for the Zeeman shift need to be multiplied by the $m_F$ quantum number.
Figure A.3: Rubidium D$_2$ transitions from the lower hyperfine level of the ground state. Indicated are the transition frequencies, information about the levels and below the corresponding saturated absorption spectroscopy signal.
$^{87}\text{Rb physical properties}$
Ringraziamenti

Innanzi tutto ringrazio il Prof. Guglielmo M. Tino che mi ha dato prima la possibilità di imparare durante il periodo della tesi di laurea e poi, non contento, di portare avanti l’esperimento durante questi tre anni di dottorato. Con la sua tenacia è riuscito a rendere MAGIA un esperimento famoso nel mondo e confido possa lasciare traccia anche per i risultati che otterrà. Molti sono i compagni di laboratorio che si sono alternati in questi anni alla ricerca di quell’agognato valore di $G$. Ricorderò la serietà e il rigore con cui Jürgen affrontava i problemi, le piacevoli giornate trascorse in lab con Marco e Torsten, anche se al tempo non facevamo altro che rincorrere le ottiche del sistema laser che se ne andavano con i dieci gradi quotidiani di escursione termica (sapere com’è cambiata la vita con il condizionatore!). Grazie a Luigi che, nonostante il suo prestigioso lavoro nelle basse terre del nord, ogni tanto è comparso in lab per darsi una mano. Ma più d’ogni altro devo ringraziare Marco e Andrea, i due lavoratori, entrambi estremamente competenti e capaci. Grazie per aver insistito nel procedere con ordine, logico e ambientale, nel lab, senza cedere alle mie tentazioni di fare misure su misure all’ultimo momento. Senza di voi non avremmo potuto ottenere questi bei risultati. Grazie per avermi insegnato molte cose e anche per aver provato a insegnarmene altre… Grazie a Marella, Antonio e Fiodor per la compagnia, le prelibatezze culinarie e le perle dialettali del napoletano.

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Ringrazio i miei più cari amici. Quelli più importanti, quelli con cui ho trascorso i momenti più belli, quelli di cui si sente la mancanza quando siamo lontani... Jacopone per primo (rullo di tamburi), praticamente un fratello, che riesce a rallegrarmi e a rassicurarmi in ogni momento con il suo spirito burlone; l’Antonia, che mi fa tanto ridere con le sue incredibili sventure, le quali la rendono comunque sempre più splendida; la Costanza una delle menti al contempo più creative e sensibili per non parlare del ritmo intrinseco nel ballare; la Francesca che mi capisce sempre al volo (e spesso non è facile), la dolce Elenina per i suoi tentativi di riunirci ancora tutti insieme; la Stefi con il suo animo colorato; Giacomo, Andrea, Davide e tutti gli altri amici universitari. Ovviamente grazie alla Cri e alla Vale, che dopo tanti anni mi sono sempre vicine e riescono ancora a ridere delle mie battute (prima o poi vi farò vedere gli atomini freddi ad occhio nudo). Ringrazio la Cinzia che dopo molti anni di lontananza è tornata regalandomi l’onore di farle da testimone di matrimonio. Grazie a tutti quelli che rendono l’atmosfera dei tornei di Mah Jong piacevole e rilassante, in particolare grazie alle citte valdarnesi. Non voglio tralasciare i nuovi arrivi nel cerchio delle migliori amicizie, Niccolò, Jacopo e le ragazze della Meeting-Volley, vecchie e move.

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